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An Educational Note On The 1D Heat Equation

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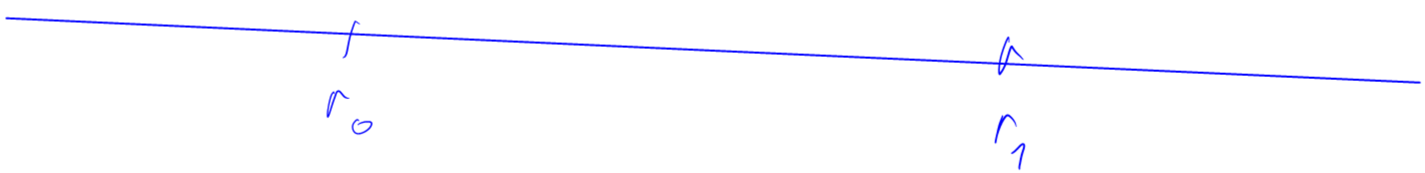
Problem

Heat equation

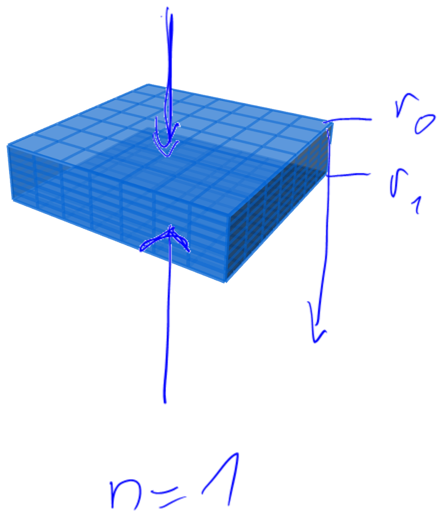
$$-k\frac{\mathrm{d}^2}{\mathrm{d}r^2}T(r) = \dot{q}(r);\, 0 \leqslant r_0 \leqslant r \leqslant r_1$$

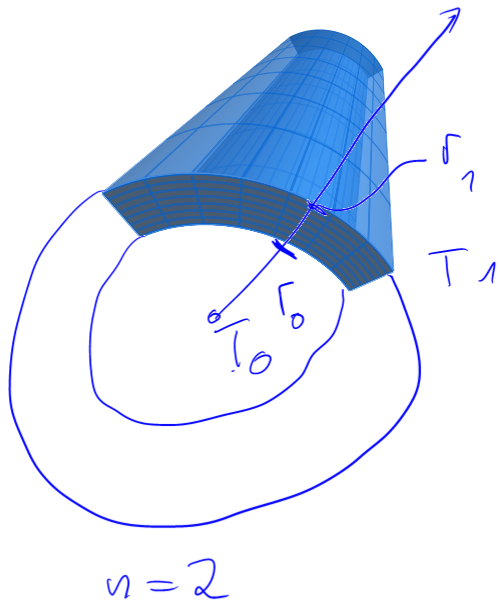
Dirichlet boundary conditions:

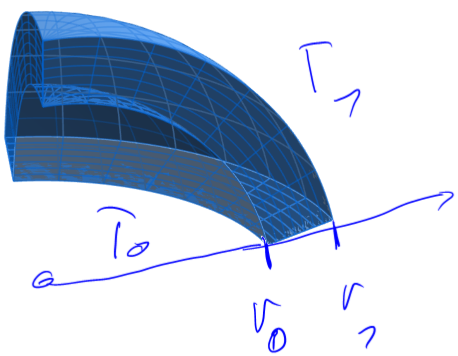
$$T(r_0) = T_0 \text{ and } T(r_1) = T_1$$



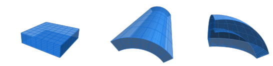
Different geometries







Different geometries



Heat equation

$$k\frac{\mathrm{d}}{\mathrm{d}r}\left\{r^{n-1}\frac{\mathrm{d}}{\mathrm{d}r}T(r)\right\} = \dot{q}(r); 0 < r_0 \leqslant r \leqslant r_1,\ n = 1, 2, 3$$

Dirichlet boundary conditions:

$$T(r_0) = T_0 \text{ and } T(r_1) = T_1$$

Linear Operator

Equation rewritten (1)

$$-k\,(n-1)\,r^{n-2}\frac{\mathrm{d}}{\mathrm{d}r}T(r) - kr^{n-1}\frac{\mathrm{d}^2}{\mathrm{d}r^2}T(r) = \dot{q}(r); 0 < r_0 \leqslant r \leqslant r_1,\ n = 1, 2, 3$$

Linear Operator

Equation rewritten (1)

$$-k\,(n-1)\,r^{n-2}\frac{\mathrm{d}}{\mathrm{d}r}T(r)-kr^{n-1}\frac{\mathrm{d}^2}{\mathrm{d}r^2}T(r)=\dot{q}(r);0<r_0\leqslant r\leqslant r_1,\ n=1,2,3$$

Equation rewritten (2)

$$-k\left((n-1)\,r^{n-2}D+r^{n-1}D^2\right)[T]=\dot{q}$$

Linear Operator

Equation rewritten (1)

$$-k\,(n-1)\,r^{n-2}\frac{\mathrm{d}}{\mathrm{d}r}T(r) - kr^{n-1}\frac{\mathrm{d}^2}{\mathrm{d}r^2}T(r) = \dot{q}(r); 0 < r_0 \leqslant r \leqslant r_1,\ n = 1, 2, 3$$

Equation rewritten (2)

$$-k\left((n-1)\,r^{n-2}D + r^{n-1}D^2\right)[T] = \dot{q}$$

Equation rewritten (3)

$$\mathcal{L}[T] = \dot{q}$$

with

$$\mathcal{L} := -k\left((n-1)\,r^{n-2}D + r^{n-1}D^2\right)$$

Linear Mappings

Linear Algebra

$$\begin{aligned} A : \mathbb{R}^n &\to \mathbb{R}^m \\ \mathbf{x} &\mapsto A\mathbf{x} \end{aligned}$$

$\ker(A)$

$$\ker(A) := \{\mathbf{x} \in \mathbb{R} \; : \; A\mathbf{x} = 0\}$$

Linear Operator

Homogeneous Equation

$$\mathcal{L}[T] = 0$$

with

$$\mathcal{L} := -k\left((n-1)\, r^{n-2} D + r^{n-1} D^2\right) \qquad \ker(\mathcal{L})$$

Well-posed for which n?

Discuss for which n this formulation makes sense?

Linear Operator

Homogeneous Equation

$$\mathcal{L}[T] = 0$$

with

$$\mathcal{L} := -k\left((n-1)\, r^{n-2} D + r^{n-1} D^2\right)$$

Well-posed for which n?

Discuss for which n this formulation makes sense?

Transfer

$$\ker(\mathcal{L}) := \left\{ f \in C^2([r_1, r_2]) \;:\; \mathcal{L}[f] = 0 \right\}$$

fundamental system

The Kernel – $n = 1$

Homogeneous Equation

$$\mathcal{L}[T] = -kD^2T = 0$$

Discussion

- What do you already know about the solution?

The Kernel – $n = 1$

Homogeneous Equation

$$\mathcal{L}[T] = -kD^2 T = 0$$

Discussion

- What do you already know about the solution?

Legibly written

We have

$$-kT''(r) = 0.$$

The Kernel – $n = 1$

Homogeneous Equation

$$\mathcal{L}[T] = -kD^2 T = 0$$

Discussion

- What do you already know about the solution?

Legibly written

We have

$$-kT''(r) = 0.$$

Discussion

- What do we know about T from this equation?
- Guess the fundamental system.

The Kernel – $n = 2$

Homogeneous Equation

$$\mathcal{L}[T] = -k\left(DT + rD^2T\right) = 0$$
$$-k\left(T'(r) + rT''\right)(r) = 0$$

Discussion

- Can we guess the fundamental system again?
- What about constants?

$T' + rT'' = 0$

Playing around?

Properties of the solution

$$T'(r) = -rT''(r)$$

$u(r)$

$\frac{1}{r}$

Playing around?

Properties of the solution

$$T'(r) = -rT''$$

Discussion

- Which type of functions fulfills

$$f^{(k)}(x) = cxf^{(k+1)}(x), c \in \mathbb{Z}?$$

- Or starting with

$$f(x) = cxf'(x), c \in \mathbb{Z}?$$

- Find an expression for the kth derivative of $\ln$, $\ln^{(k)}$.
- Try to express the kth derivative of $\ln$ by the $(k+1)$st derivative

The Kernel – $n = 3$

Homogeneous Equation

$$\begin{aligned}\mathcal{L}[T] = -k\left(2rDT + r^2D^2T\right) &= 0\\ -k\left(2rT'(r) + r^2T''\right)(r) &= 0\end{aligned}$$

Discussion

- Can we guess the fundamental system again?
- What about constants?

Further topics

Solving the boundary problem

$$\left(\begin{array}{cc|c} 1 & f_1(r_0) & T_0 \\ 1 & f_2(r_1) & T_1 \end{array}\right)$$

$\{1, x\}$

$\{1, \ln(x)\}$

$\{1, \frac{1}{x}\}$

Further topics

Solving the boundary problem

$$\left(\begin{array}{cc|c} 1 & f_1(r_0) & T_0 \\ 1 & f_2(r_1) & T_1 \end{array}\right)$$

Inhomogeneous ODE

Solving the ODE with a constant right hand side.

Thanks!

ODE-Type

If you know more application examples, with ODE of Eulerian type, please let me know.