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Introduction

The Steering committee of the SEFI Special Interest Group for Mathematics in Engineering Education (MSIG), formerly the SEFI Mathematics Working Group (MWG), is proud to present the proceedings of the 20th Seminar on Mathematics in Engineering Education organized by the Faculty of Engineering and Science, University of Agder - Kristiansand, Norway, on June 17 – 18, 2021. The Seminar was held, following a one year delay, as an on-line event, due to worldwide pandemic caused by Covid 19.

The aims of SEFI MWG (now MSIG), established in 1982, although set 40 years ago, remain relevant and up-to date:

- to provide a forum for the exchange of views and ideas among those interested in engineering mathematics
- to promote a fuller understanding of the role of mathematics in the engineering curriculum and its relevance to industrial needs
- to foster cooperation in the development of courses and support material
- to recognise and promote the role of mathematics in continuing education of engineers in collaboration with industry.

19 seminars on mathematics in engineering education have been organised by the SEFI MWG since 1984, to fulfil these aims and maintain international cooperation. The current 20th seminar is the event organised in a most difficult situation and in unusual way, at distance, in the form of an online meeting with presentations and discussion sessions in smaller groups.

Main seminar topics included traditional themes: Mathematical competencies in practice and didactical research; How to assess competencies; The goal of teaching; and a new one which arose as a result of a pandemic situation and the urgent need for distance education, online strategies and e-learning- Subjects related to teaching adaptation for COVID-19.

The programme of the Seminar included two plenary keynote lectures presented by invited speakers from Agder Univerity. Professor Simon Goodchild (director of MatRIC, Centre for Research Innovation and Coordination of Mathematics Teaching, Norwegian national centre for excellence in higher education) spoke about transformed and improved learning experiences of mathematics in engineering programmes. Professor Michael Rygaard Hansen (professor in mechatronics and Dean of the Faculty of Engineering and Science at the University of Agder) presented a talk about mathematical modelling in mechanical engineering.

SEFI MWG seminars were traditionally focused on guided discussions among participants during special discussion sessions. The proposed topic this time reflected urgent need of maths practitioners:"What did we learn from the Covid-19 experience for improving future teaching and learning?"

Good response to the seminar call for papers, represented by 14 accepted high quality papers with direct relevance to the seminar themes, resulted in a very stimulating programme including 27 presentations related to important topics in mathematical education of engineering students. All accepted contributions are included as full papers in the proceedings that are freely available at the SEFI MSIG webpage. SEFI MSIG's main objective is to provide a summary of the topics dealt with at the seminars and give free access to presented papers to all interested parties., This is in accordance with the group´s goal to gather all published materials and reports related to identified important topics in mathematical education of engineers for building up a sound body of knowledge in this field.

Finally, the author would like to thank all members of the SEFI MSIG Steering committee and the local organizers for doing the language check and editing of the proceedings for the benefit of all potential readers.

In Bratislava, June 2021

Daniela Velichová
SEFI MSIG chair

Probability and Statistical Methods: Assessing Knowledge and Competencies (pre and during Covid19 pandemic) – case study at ISEC

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Abstract

The concepts taught during a Statistical Methods course make use of different mathematical skills and competencies. The idea of presenting a real problem to students and expect them to solve it from beginning to end is, for them, a harder task then just to obtain the value of a probability given a known distribution. Much has been said about teaching mathematics related to day life problems. In fact, we all seem to agree that this is the way for students to get acquainted of the importance of the contents that are taught and how they may be applied in the real world. The study presented in this paper reflects an experience performed with second year students of Mechanical Engineering graduation of Coimbra Institute of Engineering, where the authors assessed Statistical Methods contents taught during the first semesters of 2017/2018 till 2020/2021 academic years.

During the world pandemic of Covid 19, the need to transform teaching and learning methods was urgent and their impact on evaluation was also felt. As a result, this paper also reflects the necessary adjustments on assessment that were proposed to students in the academic year of 202/2021, always having in mind the outcomes of Rules_Math project ([7]).

Keywords: Assessment, Significant Learning, Competencies, Mathematics, Engineering

Mathematics Subject Classification: Primary 46N30, 47N30; Secondary 97B40, 97B50, 97C40, 97C50

Introduction

Teaching mathematics related to day life problems has been subject of reflection for some years. In fact, we teachers all seem to agree that this is the way for students to get acquainted of the importance of the contents that are taught and how they may be applied in the real world. Nevertheless, when a student observes a real-life problem and must solve it, there are some competencies that teachers should be prepared to assess at the same time he assesses the correctness of the solution.

The definition of mathematical competence as was given by Niss ([2]) means the ability to understand, judge, do, and use mathematics in a variety of intra– and extra – mathematical contexts and situations in which mathematics plays or could play a role. Necessarily, but certainly not sufficient, prerequisites for mathematical competence are lots of factual knowledge and technical skills, in the same way as vocabulary, orthography, and grammar are necessary but not sufficient prerequisites for literacy. In the OEDC PISA document (OECD, 2009), it can be found other possibility of understanding competency which is: reproduction, i.e, the ability to reproduce activities that were trained before; connections, i.e, to combine known knowledge from different contexts and apply them do different situations; and reflection, i.e, to be able to look at a problem in all sorts of fields and relate it to known theories that will help to solve it. The competencies that were identified in the KOM project ([2], [3]) together with the three “clusters” described in the OECD document referred above were considered and adopted will slightly modifications by the SEFI MWG (European Society for Engineering Education), in the Report of the Mathematics Working Group ([1]). At Statistical Methods courses often students say that in assessment questions or exercises performed during classes the major difficulty is to understand what is asked, meaning the ability to read and comprehend the problem and to translate it into mathematical language.

The study presented in this paper reflects an experience performed with second year students of Mechanical Engineering graduation of Coimbra Institute of Engineering, where the authors assessed statistical methods contents taught during the first semesters of 2017/2018 till 2020/2021 academic years following conclusions obtain during the execution of RULES_MATH. The questions assessment tests were separated, further details may be found in ([6]), into two types: ones that referred only to problem comprehension and its translation into what needed to be calculated and others where students needed only to apply mathematical techniques to obtain the results. This paper is one of the results of RULES_MATH project, [https://rules-math.com/], which main objective was to develop assessment standards for a competencies-based teaching-learning system for mathematics in engineering education. The aims of the project can be summarized as: 1) To develop a collaborative, comprehensive and accessible competencies-based assessment model for mathematics in engineering context; 2) To elaborate and collect the resources and materials needed to devise competencies-based assessment courses; 3) To disseminate the model to European Higher Education Institutions through the partner networks and promote the dissemination all over Europe.

The innovative idea of this project was to build the curriculum on the concept of mathematics competence. Niss and his colleagues developed a framework where eight clear and distinct mathematics competencies were distinguished: thinking mathematically, reasoning mathematically, posing and solving mathematical problems, modelling mathematically, representing mathematical entities, handling mathematical symbols and formalism, communicating in, with, and about mathematics and making use of aids and tools. Although the referred competencies may overlap a bit in terms of required abilities, each competence maintains a unique major focus, a distinct “centre of gravity” ([2], p. 9). The RULES_MATH project partners’ working groups have developed a set of “Guide for a Problem” in the different areas of Mathematics that are intended to provide some examples of proposed forms of assessment and competence-based activities. The materials are available at https://rules-math.com/ and all project partners applied them to different students from different courses at their institutions. The "Guide for a Problem" presented in this paper is the one developed at IPC/ISEC for Statistical Methods.

As a consequence of the world pandemic of Covid 19, all classes were given online, and the assessment was performed in a different way. Students were asked to solve multiple choice question(s) at each class as a way of keeping them focused on the contents that were being presented and to improve students ‘participation. At the end of the semester a test was performed, and their final grade was the summation of the continuous assessment and the test grade. Conceptually the questions, either of continuous assessment or test, were designed observing the same principles followed in the previous years (separating as much as possible the competencies that were being evaluated). One major issue with online assessment was the high possibility of cheating those students had since no proctoring was made to their screens and, due to the high number of students at each class and internet difficulties, some of them had moments where their cameras and microphones were not turned on. On the following sections, we will present the conducted study and draw conclusions that were retrieved from it.

Methodology

The students that participated in the study were from the second-year degree of Mechanical Engineering engaged in Probability and Statistical Methods curricular unit. Each year a mean of 120 students engaged in this curricular unit, most of them male students. To motivate students to really engage themselves in the learning process, actual newspaper news, as shown in Figure 1. and 2., were discussed with them to show that contents were necessary to be an effective news reader and a critic person towards some articles that are object of public reading. Other real-life problems directly related with Mechanical Engineering passive to be solved using Probability and Statistical Methods were presented and discussed with students which were also invited to search and bring world situations to class and discuss them with their peers.

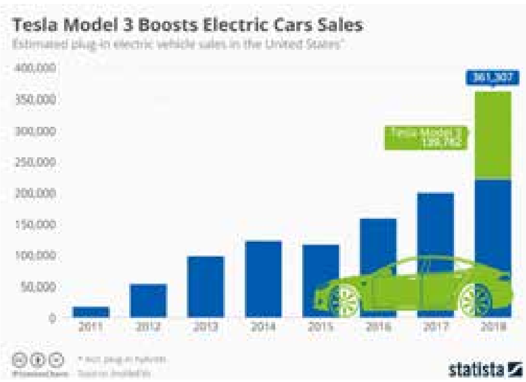

Figure 1: Newspaper news

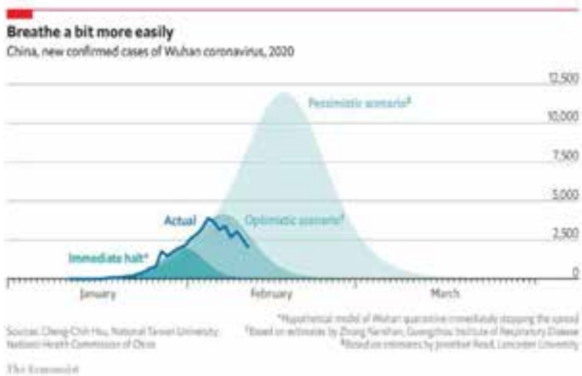

Figure 2: Covid-19 distribution - Internet information during Covid-19 pandemic

Prior to the RULES_MATH project, the exercises posed to students in assessement actvities had some of the competencies enumerated by Niss ([3]) and later worked by the SEFI Mathematic Working Group ([1]). What the authors consider that might be the reason for comments like *"We have a lot of trouble trying to understand the problem and what teachers want us to calculate!"* made sometimes by our students.

As a result of the investigation made during RULES_MATH project, a new approach to assessment was performed and the questions were, as much as possible, separated into calculus items and models identification and deduction items, and inside each one of those questions 'groups, the competencies to be assessed were as much as possible also separated, [2]. To each question a matching table like the one presented at Figure 3. was constructed to evaluate which competencies were or were not acquired by students and where more work should be done, together by students and teachers, in order to clearly acquire statistical methods competence.

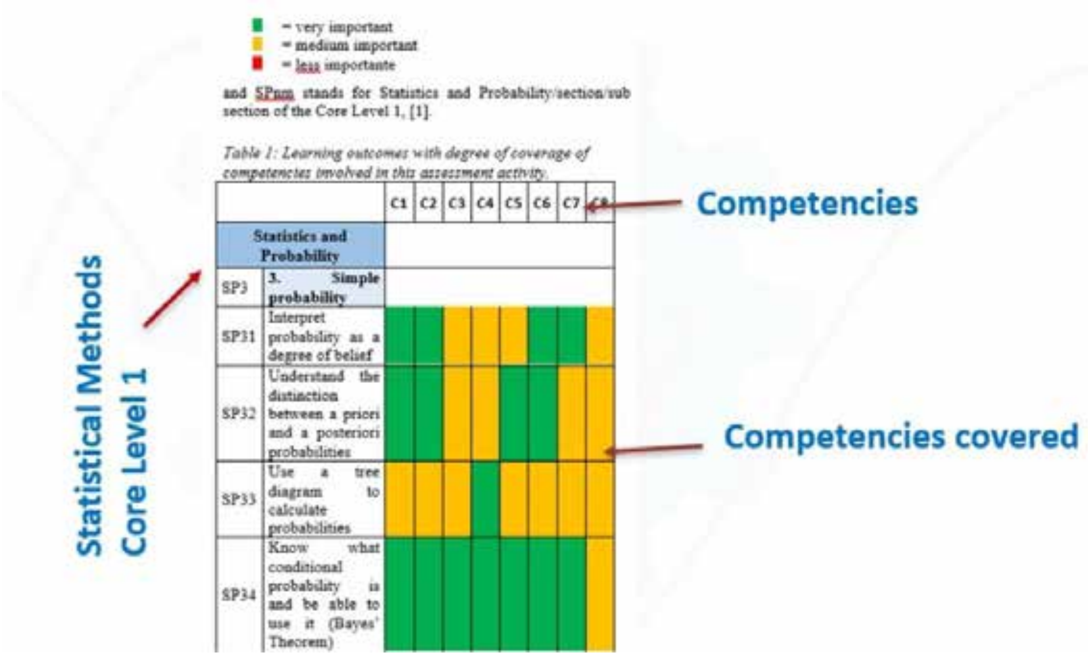

Figure 3: Learning outcomes with degree of coverage of competencies involved in this assessment activity.

From 2017/2018 until 2019/2020 the students ‘grades were obtained from activities and from two tests performed with them and in the present, 2020/2021 academic year as was mentioned on the previous section, the grade was obtained as summation of continuous assessment and a test grade. Considering the different questions groups explained above and afterwards dividing them into 2 major groups of questions, one regarding calculus and other regarding modelling questions we observed, as expected, that students have more difficulties on the modelling questions. The calculus questions are certainly more easily workable based on mimic, and they do not involve the needed literacy. Regarding authors initial preoccupation about cheating on online assessment, although some cases have been noticed there was not much to worry about. It is important nevertheless to refer that on online assessment, 6 to 10 similar questions were designed corresponding to only one question, that is we had a data base of question where each of them had 6 to 10 variants. Those variants were based on small differences that would not imply on difficult degree but had to be considered to correctly obtain the problem solution. The students performed the assessment using Moodle platform together with ZOOM platform and they had to be connected to both at the same time. The amount of work that needs to be done when preparing online assessment is effectively much more than with presential assessment.

Findings, Discussion and Future Work

Comparing grades obtained through the 3 consecutive years (from 2017 to 2020), we concluded that final grades globally are not that different although the maximum grade is higher than it used to be. We observed that 76.3% of students present at the assessment moments, had above 10 out of 20, which ensures students a good percentage course approval. In spite the referred just before, the authors believe that competence-methodology empowers students with a better preparation to face and deal with real-life problems, students became more critics and analysed questions and solutions in a more professional way.

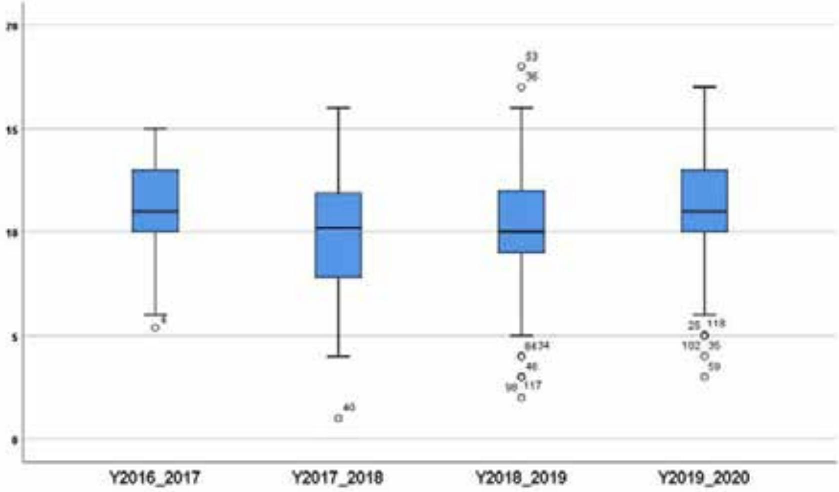

Figure 4: Grade's comparison 2017/2018/2019/2020.

Another conclusion that brought great satisfaction to the authors, see Table 1., was that students are more engaged in the exams. Throughout the past decade we observed that students were increasing their absence to assessment moments and that fact was a seriously preoccupation. With competence-methodology we observe a generous setback on this situation (the 2019/2020 year is not an example for this observation since we faced the Covid-19 pandemic which spoil the second test calm performance).

Table 1: Assessment taken by students along the study.

	Valid N	Valid Percent	Missing N	Missing Percent	Total N	Total Percent
Y2016/2017	53	44.2%	67	55.8%	120	100%
Y2017/2018	57	47.5%	63	52.5%	120	100%
Y2018/2019	85	70.8%	35	29.2%	120	100%
Y2019/2020	72	60.0%	48	40.0%	120	100%
Y2020/2021	102	85%	18	15%	120	100%

From the results 'observation until 2020 (2nd semester) we concluded that to know which competencies are less acquired by students and immediately solve those issues, we had to make a continuous assessment probably during classes what turned out to be effectively performed in 2020/2021 academic year.

Comparing the grades between 2019/2020 and 2020/2021, shown on Figure 5.) we may observe that some improvements were achieved on continuous and appeal exam but not on normal exam. In the case of the normal exam, demanding degree of difficulty was considered the same however the results were worse (maybe be attributed to the amount of work students had with other course units). The authors also concluded that with the correct assessment preparation no big worries with cheating are needed. Effectively we are teachers not security professionals, thus our function must be to teach.

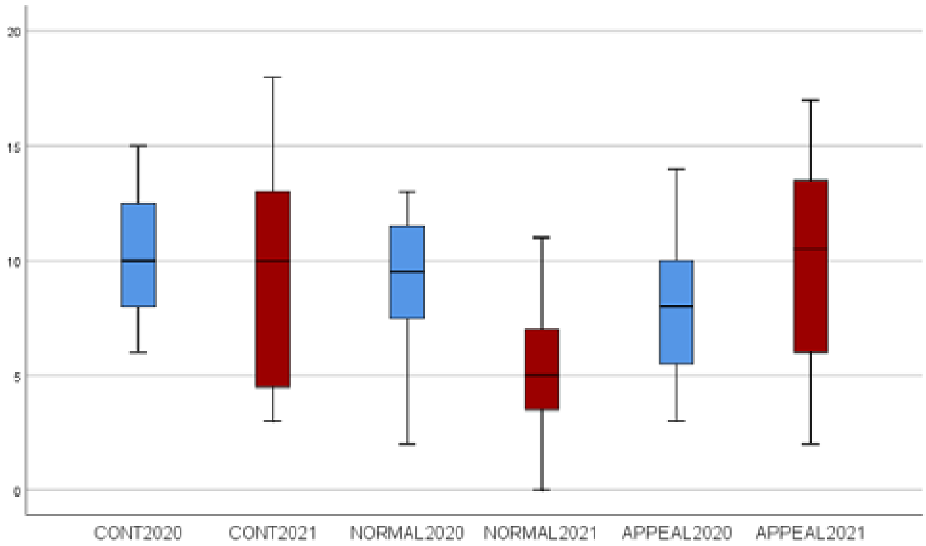

Figure 5: Results comparison 2019/2020 – 2020/2021

Other preoccupation that the authors are going to deal with next year and in the following ones, is the initial student's level up. Students at IPC/ISEC arrive from different secondary schools and different contents programs: some of them are "regular" students and others come from professional studies where the mathematical basic concepts are not dealt with the same depth as "regular" students do. Although students arriving from professional studies are, in general, more able to discuss and criticise results, problem solving techniques and even some concepts and notation are missing. As an ideal each student should be regarded as a person with his learning style and his objectives to achieve. Nevertheless, the impossibility of continuous individual attention, the student center teaching and learning process together with competence-based methodology and a previous Index of Learning Styles, ([3],[4],[5] and [6]), procedure to the course students will certainly increase student's assessment results and their in-market performance.

Acknowledgement

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References

[1] Alpers, B. et al, (2013) A framework for mathematics curricula in engineering education, SEFI,. Available online at: http://sefi.htw-aalen.de/

[2] Araceli Queiruga-Dios et al, (2020) «New Rules for Assessing Mathematical Competencies: USER GUIDE», Snezhana Gocheva-Ilieva and Araceli Queiruga-Dios Editors. © Paisii Hilendarski University Publishing House.ISBN: 978-619-202-575-5. Available online at: https://rulesmath.usal.es/3-training-open-courses/

[3] Felder, R. M. (1993) "Reaching The Second Tier: Learning and teaching Styles in College Science Education". J. College Science Teaching, 23(5), 286-290

[4] Felder, R. M. (1995). A Longitudinal Study of Engineering Student Performance and Retention: IV. Instructional Methods and Student Responses to Them. Journal of Engineering Education, 84 (4), 361-367.

[5] Felder, R. M. (1996). Matters of Style". ASEE Prism, 6 (4), 18-23.

[6] Kuri, N. P. (2004). Tipos de personalidade e estilos de aprendizagem: proposições para o ensino de engenharia. Tese de Doutoramento, Centro de Ciências Exatas e Tecnologia, Universidade Federal de São Carlos, São Carlos, Brasil.

[2] Niss, M., (2003) Mathematical Competencies and the Learning of Mathematics: The Danish KOM Project., In Proceedings of the 3rd Mediterranean.

[3] Niss, M., & Højaard, T. (Eds.) (2011). Competencies and Mathematical Learning. Ideas and inspiration for the development of mathematics teaching and learning in Denmark, English Edition, Roskilde University.

[4] Niss M., Bruder R., Planas N., Turner R., Villa-Ochoa J.A. (2017) Conceptualization of the Role of Competencies, Knowing and Knowledge in Mathematics Education Research. In: Kaiser G. (eds) Proceedings of the 13th International Congress on Mathematical Education. ICME-13 Monographs. Springer, Cham

[5] Rasteiro D. D., Martinez, V. G., Caridade, C., Martin-Vaquero, J., Queiruga-Dios, A. (2018), "Changing teaching: competencies versus contents.", EDUCON 2018 - Emerging Trends and Challenges of Engineering Education". Tenerife, Spain.

[3] Rasteiro, D. and Caridade, C. (2020) "Probability and Statistical Methods: Assessing Knowledge and Competencies", 19th Conference on Applied Mathematics Aplimat 2020, Proceedings. First edition. Publishing house Spektrum STU Bratislava, 2020. ISBN 978-80-227-4983-1, pp. 907-919, 2020.

[7] RULES_MATH Project, Project Erasmus+ 2017-1-ES01-KA203-038491" New Rules for Assessing Mathematical Competencies ". https://www.researchgate.net/project/New-Rules-for-assessing-Mathematical-Competencies.

Mathematical Competence Assessment and Work in Groups

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Abstract

The paper discusses experiences with not traditional summative assessment using group work. Dealing with nowadays pandemic situation and teaching in distance, we present opinions and attitudes of our students comparing pros and cons, and moreover we deal with forms of group work in online learning rated to be effective by our students. In addition, the paper depicts the new methodology of the pilot mathematical competence assessment introduced at the Faculty of Mechanical Engineering STU in Bratislava.

Introduction

Undoubtedly, group work becomes one of the basic modern teaching and learning methods in tertiary education. Although there was written much (see e.g. Davies, (2009); Robinson, (2006)) about its implementation, forms, possibilities of assessment (mainly regarding to project work), etc., still new items of information resulting from its application within currant particular conditions appear. In presented paper, we would like to inform interested parties about aspects on two modifications we conducted at FME STU, one where group work becomes the part of summative midterm exam, and one way we tried to find in the current completely new situation, when schools stayed closed and all education had to be in distance/on line form.

Group work as a part of summative assessment process

Last school year, we experimentally involved group work into regular summative assessment process of Mathematics I, the course in the first semester of the bachelor degree. The students were informed before, and they could also train to work in preselected groups before the exam. Taking into account that working with pre-assigned persons in groups is considered by students to be very demanding (Richtarikova, (2016)), the teams were composed upon students will. All groups in a class solved same tasks, kind of which were proposed in the Guide AC5 (Letavaj, (2020)). In the figure 1 we exhibit the achieved results of 17 groups involving 86 respondents, who's results of the group work as well as the results of all other exam parts were available, and who attended at least one term of final exam part during the examination period.

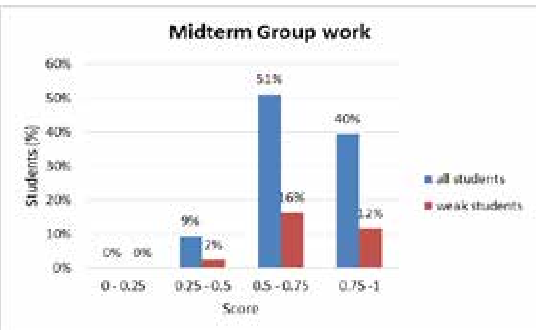


Figure 1. Group work scores

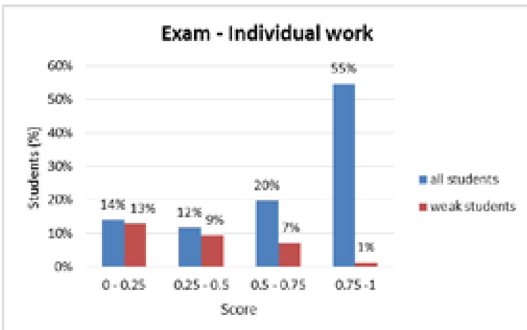
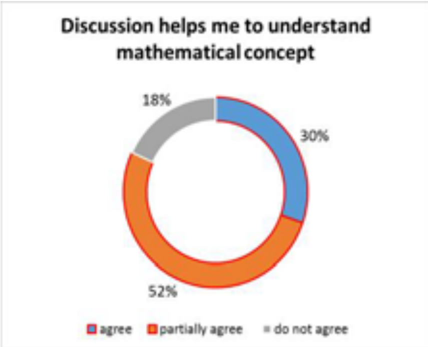


Figure 2. Exam individual work scores

Students attended daily schooling. During semester, they got over three midterm individual tests (linear algebra, function of one variable, integrals) and one more complex open task test (differentiation) elaborated in groups consisted of 4-5 members. The test was evaluated simply by the same number of points for each member of the group. The results showed very high level of success, where 91% of students achieved more than 50% of score.

Since all students had to have the same conditions for passing the course we did not use traditional experimental and control groups. In order to get a kind of evaluation, we took into consideration the results of the same students achieved in "differentiation" problems of the final individual exam part (figure 2) and those we put into comparison with "group work" results. Surprisingly, the statistical tests (Mann Whitney, Kruskal – Wallis) did not show significant difference between compared sets of results. The generally accepted premise, that results achieved in the group work are higher than results reached in individual mode was not true. It might be caused by the character of tasks, when traditionally examined basic problems on function behaviour and function properties, present in both forms, were extended in the "group work" test by two another problems including one more complex and applied one, suitable principally for solving in teams. Naturally, the impact of the "group work" test on study results of students with lower level of knowledge and skills was also in the focus. We specified the category "weak students" with results of students who achieved in three midterm individual tests the score less than 50 %; 30% of all students satisfied the rule. Here, the expected difference between compared sets of results was statistically significantly confirmed. More detailed analysis and lector's observation had shown that "weak students" formed 13% out of all 14%, who achieved score less than 25% in the individual final test. These students had low interest in study, and they quitted the study in the end. On the other hand, there were "weak" students (8% of all 75%) who managed to achieve more than 50 percent score in the individual final test, what substantially contributed to their success in the exam. Generally we can conclude that implementation of the group work into summative assessment did not negatively affect the assessment process. The score in the "group work" did not significantly differ from the score in the matching individual test. Moreover, students profited from the group work in cognitive as well as in social areas. Having their team common target, they could freely discuss and argue on relationships, procedures or results developing their mathematical competencies, what for 75% of all students implied score better than 50 percent in the matching tasks of individual final exam test (even 55% of all students had score better than 75 percent; and 20% had score between 50 - 75 percent).

Students were very surprised by integrating group work into midterm summative assessment. In the time of writing group work test, the students knew each other, and working in groups before the exam allowed them to divide the work with respect to their abilities. Teams, which were composed of lower performing students, concentrated on basic tasks. The students felt very well and useful. In survey, at the end of the semester, 82% of students claimed the discussion helped them to understand mathematical concept; to learn efficaciously, 66 % of students considered group work to be suitable, and 68% of students preferred combination of methods (figure 3). Students appreciated group work, possibility to ask questions and make discussions, the heuristic approach, comparison of various approaches and methods used in solutions. Above all, they praised the friendly atmosphere, willingness of a teacher and enjoyment.



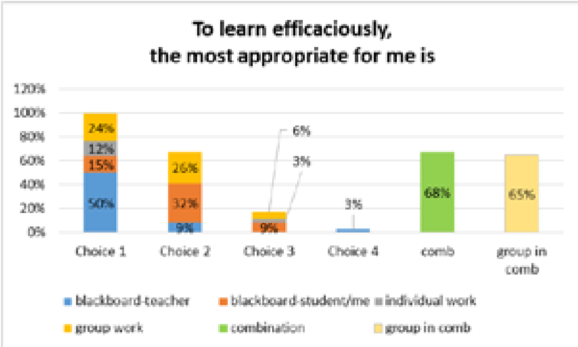


Figure 3. Two questions in survey

Group work and on-line teaching

Group work in various forms and aspects showed to be very suitable for competence oriented training, and also for especially formative assessment (Richtarikova, (2019)) of all eight particular mathematical competencies (Alpers (2013)). Switching to the distance form of education suddenly, the great pressure on self-study arose, and communication was restricted on e-mails, and if possible, on calls/group calls. After the communication platforms and shared space became available for schools, the situation has improved considerably. The quality of internet connection was (and is to today) the most limiting factor. We would like to point out some ways we brought into action in order to implement elements of group work in distance education.

1. The first attempt to bring dialogue among students did not depend on simultaneous internet connection. The students solved different problems on given topic individually, they made a record document and shared it with classmates. Each student was to check two other problems, to comment the procedure and the result. Every problem was seen at least by three students, and each student had to solve three problems. Students did not have difficulties with elaboration and simple check. What made a problem, was to argue and justify the procedure in written form. It was easier, but not easy, to comment the solutions verbally via on line meetings. The solved problems stayed available for students during all course.

2. The students speak together via voice/video channel. One participant makes notes and shares the screen of notebook. At the end the record document can be seen or downloaded by classmates.

3. The students speak together via voice/video channel. They share common interactive online whiteboard. Besides good internet connection, it requires fine hand motor skills to move the mouse – high skills in writing and drawing, or to own and master some electronic hand - writing tool. The interactive whiteboard fully substitutes the traditional blackboard. At the end the record document can be seen or downloaded by classmates.

Students worked in small 2-3 member groups, and during semester we operated all three modes mentioned above. The second way was preferred mostly in cases of explanatory calculation, or when attendees did not feel comfortable in writing by mouse, or they needed to control dataflow in connection. The group work lessons were organized in standard steps: 1. Whole class introduction; 2. Work in groups; 3. Whole class summary, presentation and discussion on group works. The first and the third part were controlled by lecturer, but the second part was operated completely under students activity. The lecturer visited the particular sessions, watched the conversation, and in cases of need entered the action and provided necessary guidance. In the background, he or she assessed the degree of students involvement, what made a part of the final evaluation. The possibility actively contribute to the joint work made the third mode very attractive and popular especially among those students who did not have problem to demonstrate their work. On contrary, the introverts and students with lower performance felt threatened but activated, being pushed out from their comfort zone. To minimize negative feelings, the activity had to be explained in the detail before. The role of the teacher was to serve as a guide and an advisor, not as an inspector or a spy. The students appreciated free friendly and safe atmosphere, where no question or answer was considered to be stupid; and in the end, teachers were surprised by live discussion, and creative productivity.

Online group work with discussions and whiteboard was considered by 63 % of students to be the teaching method which brings them the great benefit. 75 % of students felt well or very well, and only one was little dissatisfied, pointing out that not all members in the group were active, and they just took advantage from the work of others. Only 20% of students preferred individual work and 70% of students preferred combination of teaching methods. Explanatory problem solutions given by teachers were very important for 63%.

One half of respondents declared: – they like to solve problems, where thinking is required; – they like to solve problems with given procedure; – they like to argue, to give reason for their solution. One third of student claimed, they “wanted to know why”, and one third liked to solve problems in more ways. Mathematics was not favourite subject for almost 40%, but they were conscious of its importance. Almost all students spontaneously point applied problems to be the most fascinating element during the mathematics course.

The most valuable thing on online education was sharing documents and videos, and saving time by not traveling. The respondents saw the biggest disadvantage in lack of contacts with classmates and teachers, perception of a barrier and delay in communication; no ability to separate study from other life activities, presence of many disturbing elements, many hours before computer, and procrastination.

Competence assessment

Although competence indicators in the form of active verbs have already become the standard in the formulation of learning outcomes, the standardization of competence assessment still remains a challenge for the pedagogical society. Since the curriculum is very strongly tied to content, it is very difficult to move from content to competencies assessment. One of the attempts was presented in the outcomes of the Erasmus + project Rules_Math, where learning outcomes formulated by Alpers et. al. (2013) were supplemented by degree of competence importance, and by proposals of advisable tests for Core level I. The quality of student's competence was then presumed from achieved score qualitatively, in words (e.g. Caridade, et al. (2020)) or quantified through the conversion described by Gabkova, Letavaj, (2020). In addition to the mentioned, we also introduced the methodology based on direct evaluation of competencies through performance elements. The procedure could be expressed in steps:

- define the performance element representing quality of examined competence
- assign points to the elements. Scoring category is competence determined learning outputs
- calculate scores (simple or weighted) of competencies across all content related tasks
- if needed, calculate scores of tasks given by the sum of values of involved performance elements

The methodology directly provides the assessment of competencies (particular and overall) as well the assessment related to the content.

content	Performance / Competence						max
	C1 think	C2 reason	C3 probl	C4 mod	C5 repr	C6 handl	sum
multiplication	choice of repr.	reason			repr.	calcul.	4
arg, abs	concept				repr.	calcul.	3
gon. form, power			graph interpret		repr.	calcul.	3
roots cubic eq	how to grasp	reason	find the way	way	repr.		5
graph of a region	concept				repr.		2
max sum	4	2	2	1	5	3	17

Figure 4. Identification of competencies performance units and their maximal values (1/unit)

Student

content	Competence						task	relative
	C1 think	C2 reason	C3 probl	C4 mod	C5 repr	C6 handl	score	task score
multiplication	1	0			0.5	0	1.5	0.38
arg, abs	0				0	0	0	0.00
gon. form, power			1		1	1	3	1.00
roots cubic eq.	0.25	0	0.5	1	0.5		2.25	0.45
graph of a region	1				1		2	1.00
competence score	2.25	0	1.5	1	3	1	8.75	
relative competence s	0.56	0.00	0.75	1.00	0.60	0.33		**0.51**

Figure 5. Assessment of student's level of competencies through performance units

In the figure 4, we show the identification of competencies performance units and their maximal values (1/unit) in the examined test on complex numbers, where the first six competencies were assessed. The achieved competence level was evaluated by five degrees from 0 – fail to 1- excellent. In the figure 5, we demonstrate the competence assessment results of one of the students. The score of involved competencies is seen in last two rows, and the score of content elements/tasks is seen in last two columns. His or her overall mathematical competence related to complex numbers is evaluated by 0.51. Looking at the results one can easily recognize that the student did not perform any valuated item of reasoning, and he or she had considerable gaps in computations – handling with mathematical symbols and formalism. Analysing the results of all students in a class, the teacher can efficiently reveal particular competence shortcomings and the appropriate activities could be operationally included in following curricula topics.

References

Alpers, B. et al. (2013) *A framework for mathematics curricula in engineering education*, SEFI, Brussels. ISBN: 978-2-87352-007-6. Available online at: http://sefi.htw-aalen.de/

Caridade C.M.R., et al. (2020), Mathematics for Engineers: Assessing Knowledge and Competencies. In *Proc. of the 19th Conference on Applied Mathematics Aplimat 2020*, Spektrum STU Bratislava. pp. 199-211. ISBN 978-80-227-4983-1.

Davies, W. M. (2009) *Group work as a form of assessment: common problems and recommended solutions*, Springer Science+Business Media B.V. High Educ (2009) 58:563–584 DOI 10.1007/s10734-009-9216-y

Gabkova, J., Letavaj, P. (2020) The Way of Assessment of Mathematical Competencies in Rules_Math Project. In *ICMASE 2020, Proceedings of the international conference on mathematics and its applications in science and engineering*, https://doi.org/10.14201/0AQ0302

Letavaj, P. (2020) Guide for a Problem Differentiation AC 5. In Gocheva-Ilieva, S., Queiruga-Dios, A. eds. *New Rules for Assessing Mathematical Competencies – User Guide*,. Paisii Hilendarski University Publishing House. ISBN: 978-619-202-575-5, 71-87

Richtarikova, D. (2016) Enhancing ability to identify and use mathematical concepts. In *Proceedings of the 18th SEFI Mathematics Working Group seminar*. Brussels: European Society for Engineering Education. ISBN 978-2-87352-013-7, 208-212

Richtarikova, D. (2019) Training and Assessment with Respect to Competencies – Forms. In *Aplimat 2019, Proc. of the 18th Conference on Applied Mathematics Aplimat 2019*. Publishing house Spektrum STU. ISBN 978-80-227-4884-1. 989-991

Richtarikova, D. (2020) Mathematical Competency Oriented Assessment - Rules_Math Guides on Complex Numbers. In *ICMASE 2020, Proceedings of the international conference on mathematics and its applications in science and engineering*, https://doi.org/10.14201/0AQ0302

Robinson, C. L. (2006) Self- and Peer-Assessment in Group Work, In *13th SEFI European Seminar on Mathematics in Engineering Education*, Kongsberg, Norway.

High Quality Tasks for E-Assessment in Mathematics

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Abstract

This paper is dedicated to e-assessment in mathematics and the development of the corresponding electronic tasks. Due to its electronic nature e-assessment can be used to efficiently provide a huge number of tasks for training. This makes it a valuable tool for large classes, which are typical for mathematics as a service subject. In order to develop good tasks, one can address different aspects, e.g. the randomisation of problems under the restriction that they are comparable with regard to fairness, providing open-ended tasks to demonstrate understanding rather than guesses and the handling of errors, i.e. how to cope with mistakes made by students. When it comes to demonstrate understanding, mathematical proofs play a prominent role and we present a way how to transform them to an electronic task in form of a proof puzzle. A further aspect of tasks is their authenticity with regard to today's work environment which goes along with the application of specialized software. In this contribution we describe how to deal with each of the mentioned aspects of tasks.

Introduction

In recent years, digitalisation has been put forward in higher education. This transition did not only focus on digitisation of teaching material (e.g. digital lecture notes, videos and screencasts of lectures), but also changed perspectives for assessment in exercises as well as in exams. In this paper we review some aspects on electronic assessment in mathematics which we did experience within the last ten years at Hamburg University of Technology. In particular, we focus on combining advantages of electronic assessments for didactical as well as organisational purposes.

Electronic assessments – in short, e-assessments – are assessments or examinations which make use of information technology during the process of assessing, which we may divide into task generation, the assessment itself and the evaluation of the results. They can be used in formative scenarios like exercise classes or self-study time of students as well as in summative scenarios, i.e. in exams. E-assessment can support instructors in task generation by making use of randomisation such that a large amount similar as well as comparable tasks can be generated automatically; a feature which is in particular valuable in large classes. Moreover, the usage of computers and software packages allows much more advanced advances tasks in the sense that in view of Bloom's taxonomy (Bloom et al. (1956), Anderson and Krathwohl (2001)) high-quality problems can be posed much more easily. Further, it also allows for feedback during the assessment; one may think of graphical outputs of results as well as debugging hints in programming, which supports students to focus on the core contents of courses. During the assessment, the aid of computers helps to process the students' solutions efficiently which is especially important in large classes. Moreover, it makes assessments as well as exams independent from a particular location and time. Another aspect of e-assessment is automatic correction of tasks with or without automatic feedback, which makes the evaluation process very efficient. If the students' results are available electronically, they can then be processed easily to perform assessment analytics (Ellis (2013)).

In this paper, we show some aspects of how to pose high-quality tasks for e-assessments in mathematics taking into account the pros for the assessment process sketched above. In particular, we first focus on randomisation and explain how to handle errors when automatic correction of tasks is used. We then demonstrate how tasks on proving statements can be digitised and automatically corrected. Further, we explain how software packages can be used to increase the didactic quality of assessments or exams. We also shortly comment on some evaluation results.

Randomisation, handling of errors and automatic correction

With most e-assessment systems it is possible to easily generate a large number of randomised problems for training. Regarding electronic exams this is especially important to ensure fairness and prevent attempts of deception. However, implementing randomisation of problems in a naive way may result in quite easy as well as exceedingly difficult realisations of the same problem. Therefore, it is important to develop randomisation schemes which ensure a comparable workload and level of difficulty. This can be achieved by means of reverse engineering (Sangwin (2013, p. 42)). Instead of randomising the task directly, one randomises each computation step that is needed to solve the problem. For example, the calculation steps for solving a linear system of equations are encoded in the LU-decomposition of the associated matrix. To control the complexity of the calculation steps it is more suitable to randomise the entries of the LU-decomposition instead of a straight randomisation of the system of equations (Gallaun, Kruse and Seifert (2019, p. 20-22)).

For randomised problems it may be hard to provide a fair automatic correction. Especially it is difficult to consider follow-up errors if the students only enter their final answer. Adaptive problems can help to grade follow-up errors without dividing the problem in subtasks from the start. An electronic problem is called adaptive if it determines several subsequent tasks from the students answer (Bull and McKenna (2004, p. 76-77), Maravić Čisar et al. (2016)). A typical construction to evaluate follow-up errors is given in Figure 1.

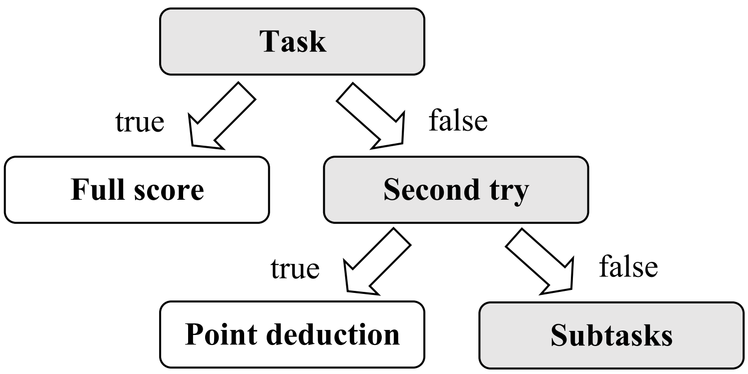

Figure 1. Adaptive task to grade follow-up errors.

First the student is given a standard task and two attempts to solve it correctly. With the second attempt the student has the possibility to check the calculations or correct typing errors for a small point deduction. If the answer is still incorrect, the student is provided with subtasks which are typically needed to solve the original task. It is now possible to check intermediate results for correctness and get partial grading. After each subtask, the solution of this step is shown, and the student can continue calculating.

Since it is generally a good strategy to generate randomised tasks through reverse engineering, intermediate answers for an adaptive problem are typically already assessable. By that the effort for creating an adaptive problem can be reduced. A drawback of providing subtasks in an adaptive problem is that one prescribes the solution approach. If the designed intermediate steps differ from the student's approach, the student may be forced to start the calculations from scratch. This can be solved by suggesting an approach in the original task, e.g. by writing "Determine an orthonormal basis by the Gram-Schmidt method". In exams were students have time pressure adaptive problems may cause problems since some extra time is needed to enter intermediate answers.

Proof puzzles

Proof puzzles are meant to assess mathematical proofs electronically (Niehaus and Faas (2013)). The students are given a statement which they should prove, using text blocks of the mathematical proof. The text blocks are randomly ordered and among them are some distractors as well. The students have to identify the right text blocks of the proof and sort them via drag 'n' drop. An example for such an exercise is the following one.

Find a proof for the following statement: "If $n \in \mathbb{Z}$ is even, then n^2 is even as well."

Then a randomly ordered list of text blocks is given and the task is to choose the right ones and to sort them in right order. The list of text blocks for the exercise above may look like:

1. $n = 2p$.
2. *This implies that* $n^2 = (2p)^2 = 2(2p^2)$.
3. $n = p^2$.
4. *This means that there is* $p \in \mathbb{Z}$ *such that*
5. *Let* $n \in \mathbb{Z}$ *be even.*
6. *Suppose that* $n \in \mathbb{Z}$ *is odd.*
7. *Hence,* n^2 *is even.*
8. *Thus, it holds that* $n^2 = 2m$ *with* $m := 2p^2 \in \mathbb{Z}$.

The evaluation of the students' solutions is done via edit distance, i.e. it is evaluated how many permutations, insertions and deletions of text blocks are needed to transform a student's answer to one of the sample solutions (Navarro (2001, p. 32)).

The advantages of such proof puzzles, besides assessing logical reasoning electronically, are that the input is quite intuitive for the student and that the correction can be done automatically. On the other hand, the fixed structure of the proof is a disadvantage because it allows for less freedom to prove the statement. Further, it is possible that a student games the system and uses the sentence structure as an aid to solve the exercise. In addition, proof puzzles are restricted to short proofs since the list of text blocks has to be concise. However, when teaching mathematics as a service subject (e.g. for engineering students) we think that this is a valid approach to implement tasks electronically which train reasoning in a mathematics context.

Usage of external software packages

A different variant of e-assessment is based on the usage of software packages for specific tasks. On the one hand, in courses on numerical analysis students may be asked to implement and test certain algorithms for solving mathematical problems (Kruse and Seifert (2018)). On the other hand, in courses on probability and statistics stochastic simulations such as simulations of random variables or (paths of) stochastic processes may help to gain the knowledge on random effects. Moreover, software tools provide the possibility to process large data sets in statistics compared to pen-and-paper tasks.

Let us focus on two particular examples. First, consider solving an initial value problem for an ordinary differential equation. In order to treat this numerically, a specific scheme has to be chosen. The choice of a good scheme as well as discretisation parameters may (and will) depend on the right-hand side of the differential equation. By using software packages, students can investigate different schemes and parameter sets and compare them graphically, as well as study convergence rates rather easily.

As a second example, take a task of hypotheses testing based on some data set and assumed underlying random process which generated the data. Then, the corresponding test scheme needs to process the data, which can easily be handled electronically. Moreover, the possibility to visualise the assumed underlying distribution and a histogram of the data provides students with some graphical feedback of their results to check for plausibility.

The usage of software packages during teaching and assessment makes the learning experience more authentic in the sense that it resembles the typical way of working in jobs nowadays. Further, it supports the students' learning process and provides a possibility to apply theoretical knowledge on realistic problems. Although these types of e-assessment neither intrinsically make use of randomisation of the tasks nor automatic correction, they can still be included in examination processes and may even make the grading process more efficient. Indeed, results may be checked by running test cases to get a first feeling of the quality of the students' solution.

Evaluation

In this article we addressed different aspects to increase the quality of tasks for e-assessment in mathematics. Each of these approaches was used in electronic exams at Hamburg University of Technology and evaluated with a survey afterwards. For example, we used electronic adaptive problems and proof puzzles in a first-year math course for engineers with about 700 examinees. In an introductory course for stochastics with 54 examinees, in addition to pen-and-paper tasks, about one third of the exam involved modelling and simulation tasks which should be solved with the programming language R in a development environment on laptops.

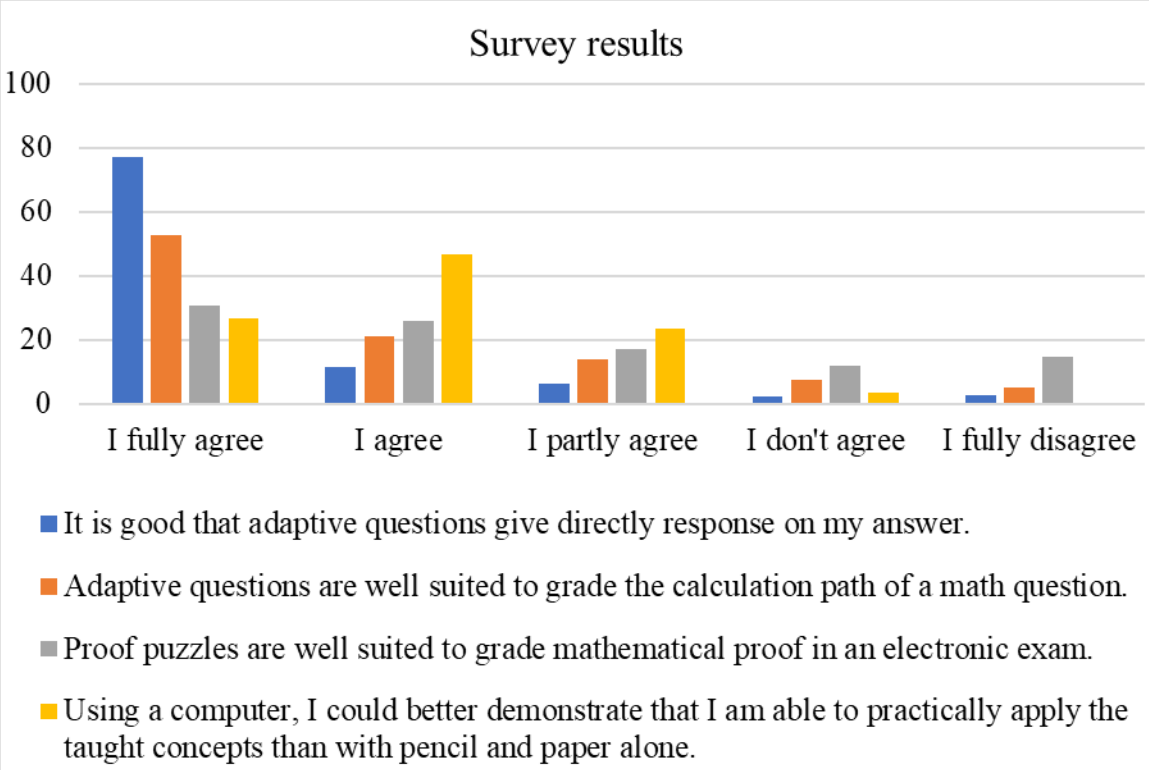

Figure 2. Results of surveys conducted in the courses Mathematics for engineers (questions 1-3, N=415) and Stochastics (question 4, N=41).

References

Anderson, L.W. and Krathwohl, D. (2001) *A Taxonomy for Learning, Teaching, and Assessing. A Revision of Bloom's Taxonomy of Educational Objectives.* New York: Longman.

Bloom, B. S., Englehart, M. D., Furst, E. J., Hill, W. H. and Krathwohl, D. R. (1956) *The Taxonomy of educational objectives, handbook I: The Cognitive domain*. New York: David McKay Co., Inc.

Bull, J. and McKenna, C. (2004) *Blueprint for computer-assisted assessment.* London: RoutledgeFalmer.

Ellis, C. (2013) Broadening the scope and increasing the usefulness of learning analytics: The case for assessment analytics. *British Journal of Educational Technology* Vol. 44 (No. 4):662-664, DOI: 10.1111/bjet.12028.

Gallaun, D., Kruse, K., and Seifert, C. (2019) "Adaptive Übungs- und Prüfungsaufgaben in Mathematik mit hochwertiger Bewertung." In D. Schott, eds. *Proc. 15. Workshop Mathematik in ingenieurwissenschaftlichen Studiengängen, Rostock-Warnemünde*. Wismar: Gottlob-Frege-Zentrum, pp. 18-24.

Kruse, K. and Seifert, C. (2018) "Implementing Computer-assisted Exams in a Course on Numerical Analysis for Engineering Students." In *Proc. of the 19th SEFI MWG*, Coimbra: The Department of Physics and Mathematics Coimbra Polytechnic - ISEC, pp. 33-38.

Navarro, G. (2001) A guided tour to approximate string matching. *ACM Computing Surveys*, 33(No. 1): 31-88, DOI: 10.1145/375360.375365.

Niehaus, E. and Faas, D. (2013) "Mathematische Beweise in elektronischen Klausuren in der Lehramtsausbildung." In G. Greefrath, F. Käpnick and M. Stein, eds. *Beiträge zum Mathematikunterricht 2013*. Münster: Institut für Didaktik der Mathematik und Informatik, Universität Münster, pp. 704-707, DOI: 10.17877/DE290R-14028.

Maravić Čisar, S., Čisar, P., and Pinter, R. (2016) Evaluation of knowledge in Object Oriented Programming course with computer adaptive tests. *Computers & Education*, 92-93: 142-160, DOI: 10.1016/j.compedu.2015.10.016.

Sangwin, C. (2013) *Computer aided assessment of mathematics.* Oxford: Oxford University Press.

Design principles for final answer assessment in linear algebra: implications for digital testing

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Abstract

Digital testing, such as multiple choice questions and final answer items, offers many advantages in higher educational assessment practices. Well designed digital grading is more reliable and faster than hand grading and is scalable to larger classes. The validity of digital grading is open to criticism, particularly in mathematics where much of mathematics is based on processes and reasoning and not on the final answer achieved. At a technical university in the Netherlands we have been increasing our use of digital short answer testing in calculus and linear algebra for service mathematics. To assess the validity of this mode of assessment we graded a linear algebra test in two ways, short answer grading (where answers were considered either correct or incorrect) and so-called "hypothetical grading", where we assigned a grade based on the fully worked solution. Certain types of items proved to be more suitable for short answer (and hence digital) testing than others. We concluded our analysis with a set of design principles for digital or short answer testing in linear algebra.

Introduction

Digital testing of mathematics holds many advantages over pencil and paper grading with its speed, opportunity for immediate automated feedback, potential for scaling up and not being susceptible to grader error (Aarts, 2018; Sangwin, 2019). Use of digital testing, or computer aided assessment (Broughton, Robinson and Hernandez-Martinez, 2013; Lawson, 2012) is on the rise as class sizes increase globally and the tools and platforms available become increasingly sophisticated (Koomen and Zoanetti, 2018; Sangwin, 2019). In this paper we are concerned over the design of good quality "final answer" items for assessment since those are amenable to digital testing.

Despite the many advantages, there are concerns surrounding final answer testing of mathematics (Lawson, 2012; McGuire, Youngson, Korabinski and McMillan, 2002). These concerns are frequently grounded in the importance of being able to assess process in any mathematical task rather than simply the production of an answer. Current digital testing can generally only assess the final answer of the question, rather than the process, with some exceptions such as embedded answer items (Martins, 2018) and freehand sketching (Yerushalmy, Nagari-Haddif, and Olsher, 2017). Designing good assessment items for final answer grading is more challenging than designing items for traditional partial credit grading. As such, design principles informing the creation of valid and reliable test items that assess the desired learning outcomes when the process is not visible are valuable.

In June 2019, a Linear Algebra test was written, where 50% of the questions were only graded on their final answer. Two of the four cohorts of students who wrote the test handed in their rough work for analysis purposes to investigate the quality of the exam, especially that of the final answer questions. The analysis discussed below provided insight allowing us to refine the existing design principles for the creation of good quality items for final answer (preferably digital) testing..

Research Questions

The analysis carried out in this study considered the validity of grading the items by their final answer only by comparing that mode of grading to a partial credit mode in order to determine the quality. The objective in this study was a set of design principles for items for digital grading, specifically final answer items and in the context of linear algebra. To support this objective, we asked two research questions:

1. What do the quantitative differences between grades achieved through final answer grading versus partial credit grading tell us about the quality of the items?
2. What types of errors are observed in students' working and how does that inform item design?

Design-based Research

Design based research consists of different phases. The model produced by McKenney and Reeves (2018) consists of three main phases: Analysis, Design and Evaluation. This paper represents part of the Evaluation phase of the cycle, having developed a first set of design principles from the linear algebra test of the previous year and research done on Calculus in the University of Twente (Lochner, 2019; Aarts 2018), and we decided to apply and evaluate these principles in this exam. For the reader, we will recall the design principles in this section, reflect on the implementation process in the methodology section and then describe the outcomes in the results section.

The design principles consisted of

- Limit the amount of points assigned to an item to a maximum of 3.
- An item should be a one step process not susceptible to careless error OR the answers should be checkable and students should have had the opportunity to learn how to do this.
- The algebra involved should be minor, testing the core concept of the question, rather than algebraic manipulation.
- As the learning goals that can be tested in this way are limited, a maximum of 50% of the grade should be based on final answer questions.

Methodology

Four cohorts of students enrolled in an introductory linear algebra course in March 2019: physics, electrical engineering, advanced technology and mechanical engineering students. The course lasted one academic quarter of ten weeks. The cohorts of physics (N=47) and electrical engineering (N=77) students were chosen to have the grading of their work analysed for this study.

The students' work was first graded for summative purposes. Half of the paper (18 of 36 points) was in the form of fully worked solutions which could receive partial credit. The other half was graded only on the final answer. The university is in the process of increasing the number of devices available for use in a digital test. Due to scheduling difficulty, in this instance all students completed the "digital" part of the test on paper, writing their answers into blocks on an answer sheet; this process was already familiar to them. The analysis discussed below therefore was of paper-proxy for digital testing which had the dual result of providing insight into final answer testing as well as the effectiveness of having a paper-proxy.

The final answer questions were re-graded using a grading scheme which allowed partial credit, hereafter termed "hypothetical grading" or "partial credit grading". (For related work see McGuire, Youngson, Korabinski and McMillan, 2002; Ashton, Beevers, Korabinski, and Youngson, 2006; Rane and MacKenzie, 2020.) We were interested not only in whether the grades differed but also in why. We differentiated between arithmetic (and copying) errors and understanding errors.

Results and Discussion

In this section we present the results of our comparison of final answer and hypothetical (partial credit) grading. We consider whether our existing design principles delivered good items and use our observations to evaluate our design principles.

All items (see Appendix) adhered to our design principles. Item 1a (computing a determinant) tests a single step process not readily susceptible to arithmetic errors. It is checkable by computing the determinant a second time, expanding about a different row or column to the first time. Item 1b (determinant of inverse) can be checked by solving the item in two ways, either the inverse can be recalculated, or it can be the reciprocal of 1a. Item 3 (checking if vectors are in a subspace) was checkable; however, this required a few steps to do so and was prone to error. Items 5a and 5b (eigenvalues and eigenspaces) were quickly checkable. As according to our design principles, students had sufficient opportunity to practice these types of questions and how to "check their answer".

We investigated what kinds of errors the students were making stopping them from getting full credit for an item in either hypothetical or final answer grading. All correlations for items 1, 3 and 5 between their final answer grading and their hypothetical grading were above .75, which indicates a strong relationship, meaning that even though there was a significant difference in the way of grading these items (partial vs final grades) that there was not a great differences in the grades. In many of the instances when students achieved more points on the hypothetical grading than in the final answer grading it was from being awarded points for setting up a problem, perhaps doing some calculation, but then not following through with correct interpretation of results. This behaviour is not in line with the learning goals and does not indicate true proficiency in Linear Algebra. A criticism often levelled by students at the "unfairness" of final answer grading is that they at least know part of the process and hence should receive part of the grade. However, with these short (maximum 3 point) questions, the solution process is short, and we observed that the majority of the time, students would hypothetically only have gained a few points for setting up the problems. This need for conceptual understanding in order to gain the full grade in final answer testing can be seen as an advantage from the point of view of the teacher. That is: no points for superficial manipulation of matrices, full grade only if true understanding is shown.

Item 4 proved a problematic item and contributed more than any of the other items to refining our design principles. Item 4 consists of two parts: 4a and 4b. Each subpart was awarded 3 points in the final answer grading mode. Given a 4x3 matrix, 4a asked the student to find a basis for the null space of the matrix, whilst 4b asked for a basis for the column space of the same matrix. The hypothetical grading was structured differently with 2 points each for correctly reducing the matrix to reduced echelon form (1 point if a minor arithmetic error), correctly interpreting the reduced matrix to extract the basis for Null A (1 point if Null A = Span{}), and for correctly interpreting the reduced matrix to extract the basis for Col A (1 point if Col A = Span{}). This grading structure of 2+2+2 is structurally different from the final answer structure of 3+3. This caused there often to be a difference in grades between the two grading schemes, making quantitative statistics difficult for this item, however much was learned.

The assumptions made in the design of this item proved problematic. Firstly, only if students wrote "a basis for Null/Col A = {(),()}" or simply "{(),()}" would they gain full points, for a or b respectively. However, since this was a paper proxy of digital testing, grader discretion played a role here as some graders were more lenient about notational variation. A further problem was the non-uniqueness of bases. 3 students column reduced instead of row reduced, resulting in a basis that did not look like the grading scheme, but was indeed correct. A consensus was reached that if a student erroneously wrote the words "span" in both answers, and was otherwise completely correct, that students only lost points for one of the sub-questions.

When Item 4 was initially designed we did not consider it a particularly problematic item, however our analysis clearly indicated it to be unsuitable for final answer testing, in particular for our paper proxy for digital testing. First, having two similar items raised the double penalty issue, where students might be penalised twice for a single error – in this case incorrect use of the term "span" (see also Ashton, Beevers, Korabinski, and Youngson, 2006). Digital testing would either penalise twice or the test would have been designed so that students needed only to enter numbers, which would then weaken the items as regards testing whether the students know the difference between a basis and the span of that basis, concepts known to be problematic (Stewart and Thomas, 2010). Secondly, bases are not unique. Graders expecting to see one answer might consider a different correct answer as incorrect. Digital testing would only avoid this issue if the system were able to use a computer algebra system to test for equivalence of such sets. Grading work by students that give a variety of correct answers, Sangwin (2019) agrees that computer algebra systems are ideal, as this saves teachers from making mistakes and a tremendous amount of checking. Thirdly, linear spaces have notation requirements not limited to use of the "span", such as correct use of set notation. There might be notation about which you as the teacher are willing to be lenient (such as commas or semicolons between vectors) and other notation you consider crucial, such as braces (curly parentheses). Hand grading allows for flexibility which a digital system might not.

Conclusion and Revised Design Principles

Grader error and variation would disappear if the test were truly tested digitally rather than using our paper-proxy system. For reliable grading consistency is important when multiple people are grading the same test. We conclude that true digital grading is more reliable than hand grading as a proxy for digital grading but that goes hand in hand with a well-designed test that does not feature the double penalty issue.

Our analysis of Item 4 provided us with two important new design principles for (digital) final answer testing. First, do not include two items that might result in penalising the same error twice. Secondly, when an item has (infinitely) many correct answers only include it if the digital system has a means of testing for equivalence.

Our design principle of "an item should be a one step process not susceptible to careless error OR the answers should be checkable" we have adjusted to: "... OR the answers should be **quickly** checkable". If it takes as long to check the answer as it did to answer the question, it renders the item unsuitable unless unlimited time has been made available for the writing of the test. Similarly, the checking process itself should be relatively immune to careless error, a characteristic not shown by Item 3, for instance.

Refined set of design principles:

- Limit the amount of points assigned to an item to a maximum of 3.
- An item should be a one step process not susceptible to careless error OR the item should be quickly checkable and students should have had the opportunity to learn how to do this.
- The algebra involved should be minor, testing the core concept of the question, rather than algebraic manipulation. Consider avoiding fractions.
- Avoid items where the same error could be penalised twice.
- Avoid items with notational complexities where (were the item to be hand graded) grader opinion might differ
- If infinitely many answers are correct (for example a basis of a subspace) the testing procedure must allow for recognition of any correct answer.
- As the learning goals that can be tested in this way are limited, a maximum of 50% of the grade should be based on final answer questions.

References

Aarts, H. (2018) A Hybrid Test for Mathematics. *Unpublished report for Senior University Teaching Qualification, University of Twente*. Available from the authors upon request. Poster presentation accessed via https://bit.ly/3wlKfqI

Ashton, H.S., Beevers, C.E., Korabinski, A.A. and Youngson, M.A. (2006) Incorporating partial credit in computer-aided assessment of Mathematics in secondary education. *British Journal of Educational Technology*, *37*(1), pp.93-119.

Broughton, S.J., Robinson, C.L. and Hernandez-Martinez, P. (2013) Lecturers' perspectives on the use of a mathematics-based computer-aided assessment system. *Teaching Mathematics and Its Applications: International Journal of the IMA*, *32*(2), pp.88-94.

Koomen, M. and Zoanetti, N., (2018) Strategic planning tools for large-scale technology-based assessments. *Assessment in Education: Principles, Policy & Practice*, *25*(2), pp.200-223.

Lawson, D. (2012) Computer-aided assessment in mathematics: Panacea or propaganda?. *International Journal of Innovation in Science and Mathematics Education*, *9*(1).

Lochner, A.J. (2019) “Summative digital testing in undergraduate mathematics : to what extent can digital testing be included in first year calculus summative exams, for Engineering students?”. *Accessed via* *https://essay.utwente.nl/77167/1/Lochner_MA_EST.pdf* (21 May 2021)

Martins, S.G. (2018) A Study of the Application of Weekly Online Quizzes in Two Courses of Mathematics for Engineering Students–Is it a fair and effective strategy to increase students’ learning?. *International Journal of Innovation in Science and Mathematics Education*, *26*(1).

McKenney, S. and Reeves, T.C. (2018) *Conducting educational design research*. Routledge.

McGuire, G.R., Youngson, M.A., Korabinski, A.A. and McMillan, D. (2002) Partial credit in mathematics exams-a comparison of traditional and CAA exams. In Proceedings of the 6th CAA Conference, Loughborough: Loughborough University.

Rane, V. and MacKenzie, C.A. (2020) Evaluating students with online testing modules in engineering economics: A comparision of student performance with online testing and with traditional assessments. *The Engineering Economist*, *65*(3), pp.213-235.

Sangwin, C. (2019, February) Developing and evaluating an online linear algebra examination for university mathematics. In *Eleventh Congress of the European Society for Research in Mathematics Education* (No. 15). Freudenthal Group; Freudenthal Institute; ERME.

Stewart, S. and Thomas, M.O. (2010) Student learning of basis, span and linear independence in linear algebra. *International Journal of Mathematical Education in Science and Technology*, *41*(2), pp.173-188.

Yerushalmy, M., Nagari-Haddif, G. and Olsher, S. (2017) Design of tasks for online assessment that supports understanding of students’ conceptions. *ZDM*, *49*(5), pp.701-716.

Appendix

Item 1
Given is the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix}.$$

a) Determine $\det(A)$.
b) Determine $\det(A^{-1})$.

Item 3
Consider a vector space and four vectors in $\mathbb{R}^3$:

$$\mathcal{T} = \text{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}\right\} \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$$

Indicate for each of these four vectors whether they are an element of $\mathcal{T}$ or not.

Item 4
Given is the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 & 2 \\ 1 & 1 & -1 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix}.$$

a) Determine a basis for Null A.
b) Determine a basis for Col A.

Item 5
The matrix A is given by

$$A = \begin{pmatrix} 1 & -3 & -1 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

a) A has eigenvalue -1 (you don't have to prove this)
   Determine the corresponding eigenspace.
b) Determine the other (possibly complex) eigenvalue(s) of A.

Mathematics for engineers: a case study about assessing knowledge and competencies

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Abstract

Assessment is seen as a vehicle for improving learning and results that allows the creation of effective conditions for everyone to learn, promoting a culture of success. Assessment through learning objectives and mathematical competencies involved in engineering education has been extensively studied by the RULES_MATH project partners. This paper intends to evaluate these criteria on complex numbers. Thus, a test on complex numbers was developed that assesses the 5 major learning objectives of this content and their association with the 9 mathematical competencies. The test was carried out by 126 students of Biomedical Engineering, Electromechanical Engineering and Mechanical Engineering from Coimbra Institute of Engineering. The results obtained were satisfactory since at almost questions (7 out of 9) students were able to obtain a positive grade. The performed assessment test also permits to identify level differences between the courses where it was applied. Relating to the test itself as a tool to assess competencies, we may conclude that it covers most of the competencies that we need to evaluate, and its difficulty is adequate to our students.

Introduction

Evaluating is a process that involves conceptions, values, principles, theories, goals, desires, trajectories, and when the process focuses on education it becomes potentially more complex, creating challenges to its verification and recording (Ross 2010). Evaluating in education should be understood, not only as a product of education and classification of students, courses, institutions, but mainly as a process with educational, pedagogical, and psychological characteristics. Educational assessment, however, consists of much more than randomly designed tests and tests designed to verify that the student has “learned the content of the Curricular Unit”. It is a complex process that, when well planned and executed, can become a powerful tool in building students’ learning.

The educational assessment process presents an interdependence between knowledge learning and competencies development (Caridade and Rasteiro 2019, Boesen et al. 2018). In this perspective, knowledge and competencies are processes that are articulated, but not confused. In fact, the use of knowledge is a strong requirement in the process of building competencies. However, the more human actions that require the deepening or organization of knowledge, the more time is required for the development of competencies. Thus, one of the major future challenges facing evaluation processes is to privilege, among the aspects to be evaluated, the development of competencies (Niss and Højaard 2011, Niss et al. 2017). According to B. Alpers et al. in the Framework for Mathematical Curricula in Engineering Education (Alpers et al. 2013), previously proposed in the Danish KOM project (Niss 2003) the mathematical competencies are:

(C1) thinking mathematically; (C2) reasoning mathematically; (C3) posing and solving mathematical problems; (C4) modelling mathematically; (C5) representing mathematical entities; (C6) handling mathematical symbols and formalism; (C7) communicating in, with, and about mathematics and (C8) Q. Thinking about this, it was proposed 'New Rules for assessing Mathematical Competencies' project to change the educational paradigm and to get a common European teaching and learning system based on mathematical competencies rather than contents [https://rules-math.com/].

The RULES_MATH (2020) project working groups have developed a set of "Guide for a Problem" in the different areas of mathematics that are intended to provide some examples of proposed forms of assessment and competence-based activities. The materials are available to all project partners to apply to different students from different courses and institutions. One such material is AC3 which aims to evaluate students about complex numbers. In the scheme presented in Figure 1, a view of the role of mathematics education in engineering can be seen. Inputs and outputs to engineering mathematics education, links between engineering education with society and industry, and the engineering mathematics education focused on learning objectives and mathematical competencies.

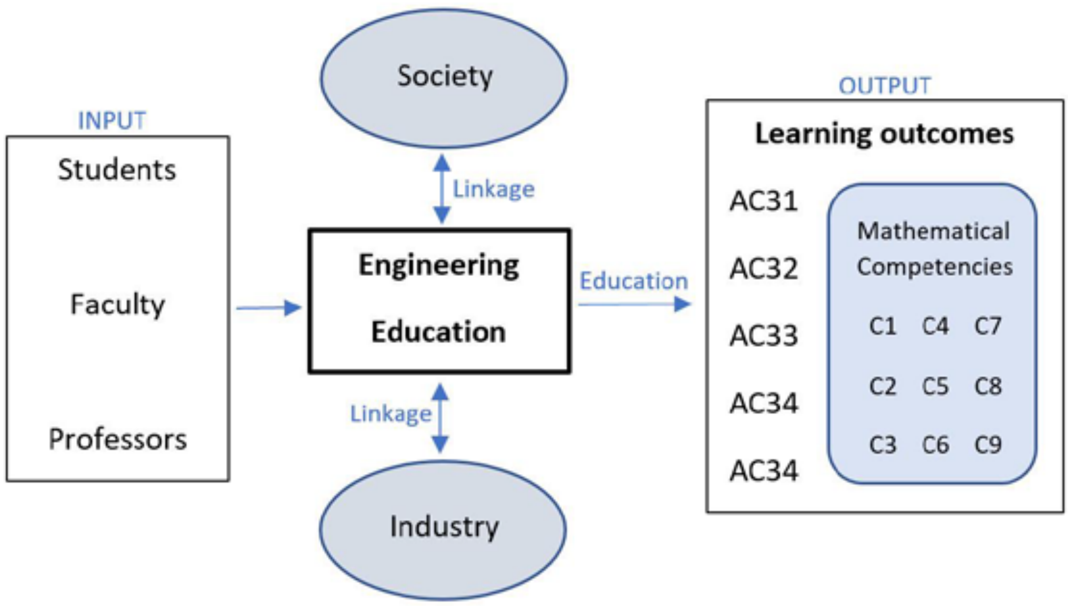

Figure 1. Schematic of inputs and outputs for Mathematical Engineering Education.

Methodology

As a way of studying mathematical competencies in teaching complex numbers, the AC3 guide was analysed (Caridade 2020). For this, an investigation methodology based on two different forms was proposed:

- Analysis of the questions presented in the test, their mathematical competencies, and their learning objectives.

- Analysis of the competencies and learning objectives acquired by the students through the questions of the test.

To obtain a relationship between the learning results and the eight mathematical competencies, the table represented in Figure 2 was elaborated. Each question was associated with the competencies inherent in the learning process and its importance using different colours (green - very important, yellow - medium important and red - less important).

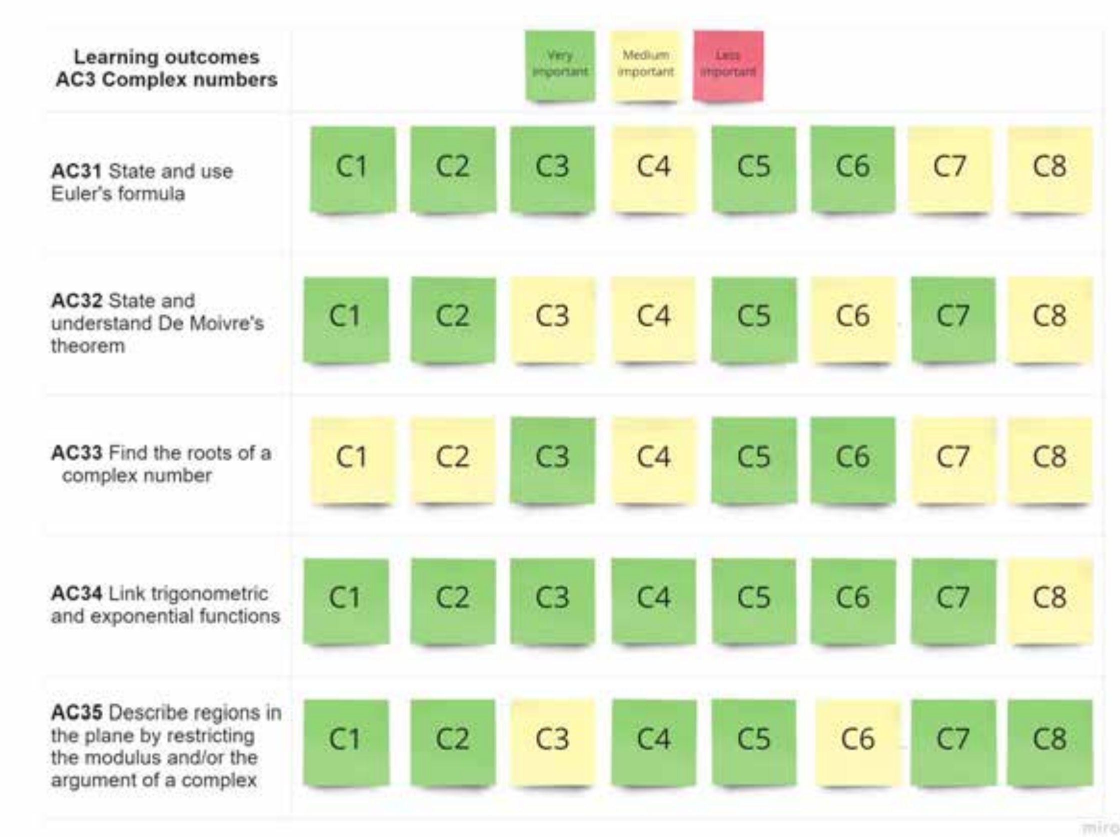

Figure 2. Learning outcomes with degree of competencies involved in this assessment activity coverage.

In the results of learning complex numbers (AC3), it appears that none of the mathematical skills involved is less important (represented in red). Most of the competencies involved in this content are covered with classification very important (represented in green) which means that the exercise proposed will provide a very good evaluation of its achievement. About 62.5% (25 out of 40) have very important coverage as competencies and 12.5% (15 out of 40) have medium importance coverage The five learning outcomes have a percentage of 62.5% (5 out of 8) for AC31, 50% (4 out of 8) for AC32, 37.5% (3 out of 8) for AC33; 87.5% (7 out of 8) for AC34 and 75% (6 out of 8) for AC35. Thus, "Link trigonometric and exponential functions" is considered as being the most important learning result in this assessment test, only the competency C8 has medium importance and "Find the roots of a complex number" is the result evaluated with least importance, with only 3 high competencies (C3, C5 and

C6). As for the presence of competencies in the learning results, it appears that the C5 competency is the one considered as the most important (present in all learning results with high importance) and the C8 competence the least important (present in only one learning result with high importance).

Case study

The assessment described in AC3 was implemented with linear algebra students, since it is in this course that a small review of the complex numbers is made, and the complex numbers in the Euler's form is presented to students for the first time. The study was carried out in a group of 126 students of Linear Algebra from Biomedical (20 students), Electromechanical (27 students) and Mechanical (79 students) Engineering in Coimbra Institute of Engineering.

Students took the same test on complex numbers with 9 questions in total (Q1, Q2, Q3, Q4, Q7, Q8, Q9). The first 4 were multiple choice questions (Q1-Q4) and the last 5 development questions (Q5-Q9). Attached is a pdf of the test performed. The 4 multiple choice questions are based on complex number definition and interpretation and simple operations. In the first and fourth question only one answer is correct (between 4) in the second and third questions there are several correct options (between 10 and 11 respectively). The developmental questions aim to better understand the student's learning on this subject. Several questions were asked about the meaning of the calculations performed, why the student’s calculations were performed in one way or another and how to solve it in a more efficient way. The contents evaluated in this part of the test are complex operations (include roots of complex); complex representation in Argan plane; complex number in exponential formula, Euler's formula and De Moivre's formula.

Figure 3 shows all the learning objectives (AC31, AC32, AC33, AC34, AC35) involved in the test of complex numbers proposed to students in each test question (Q1 to Q9). All competences involved in each of the questions within the learning objectives are also identified. Thus, for example in question Q5, within the learning objective "AC31-State and use Euler's formula", three mathematical competences are identified (C1, C5 and C6).

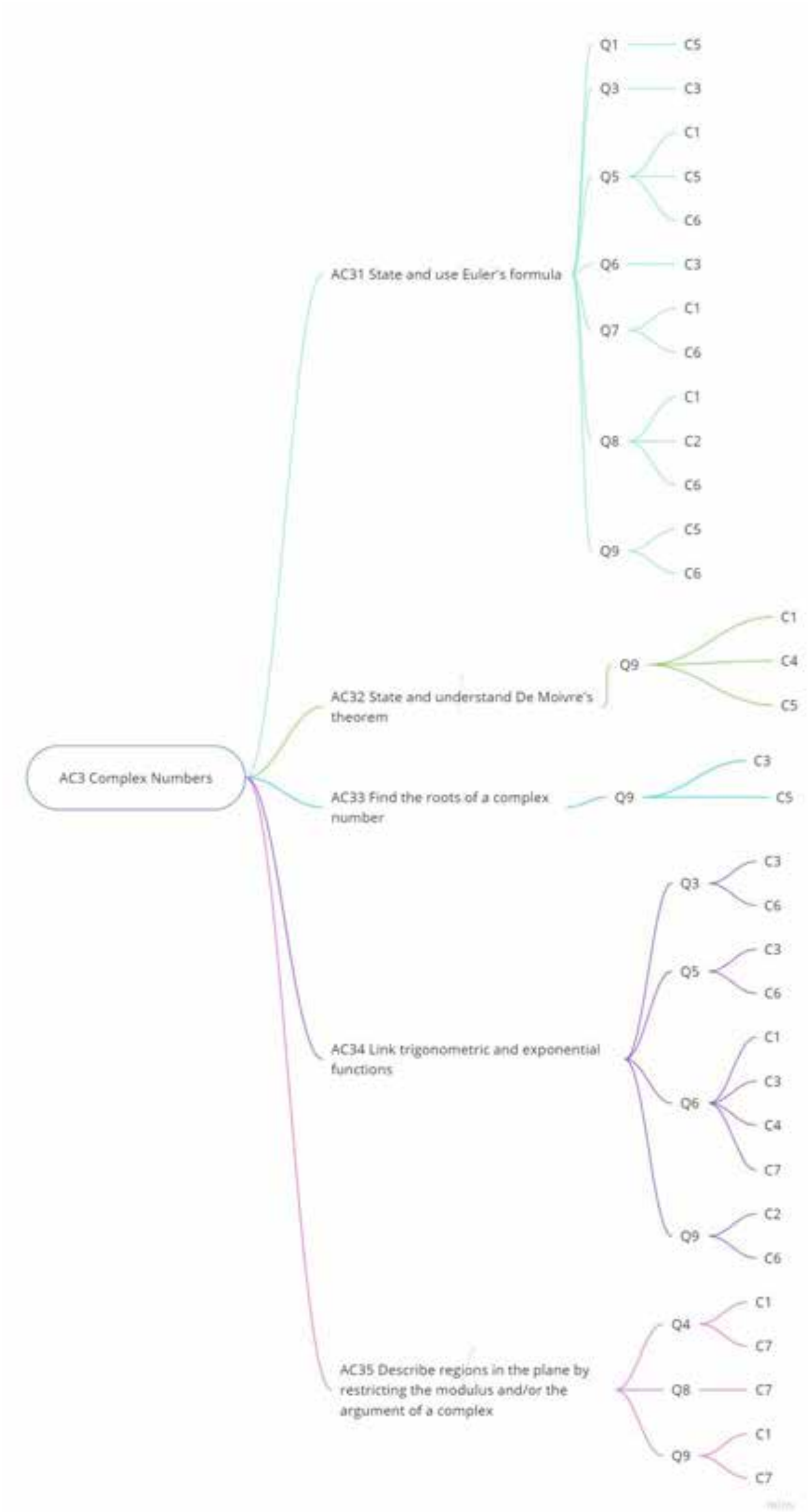

Figure 3. Learning outcomes with degree of competencies involved in this assessment activity coverage.

The most frequent learning result is the AC32 that appears in 7 of the 9 (Q1, Q3, Q5, Q6, Q7, Q8, and Q9) test questions, the least frequent is the AC32 and AC33 that appears in only one of the nine questions (Q9). Question 9 contains all the learning results assessed on the test and is therefore the most complete question, while question 1 only focuses on the AC31 learning result. Question 2 is not related to any of the most important learning outcomes and questions 4 and 7 are only related to one of the learning outcomes (AC35 and AC31 respectively). Regarding the skills involved in the different proposed questions, it appears that the skills that were most evaluated were the skills C1 and C6 (appear 7 times in the test), followed by C6 (6 times), C5 (5 times), C7 (4 times), C2 and C4 (2 times). The C8 competence ("make use of aids and tools for mathematical activity") was not evaluated in this test, since to evaluate the use of mathematical tools it is necessary that the student performs work or activities and not through a written test. As a global aspect, it is possible to verify that within each learning result, each question in most cases, only evaluates one to two competences, three competences are evaluated only in questions Q5 and Q8 of the learning result AC31 and Q9 of the AC32 result, four competences, only in question Q6, within the AC34 learning result.

Student's response and interpretation

After the students, take the test, the teacher corrects them, assigning a rating between 0 and 100 for each of the questions. The results obtained are represented in the Figure 4 to 6. The answers obtained by the 20 students of Biomedical Engineering are represented in the Figure 4.

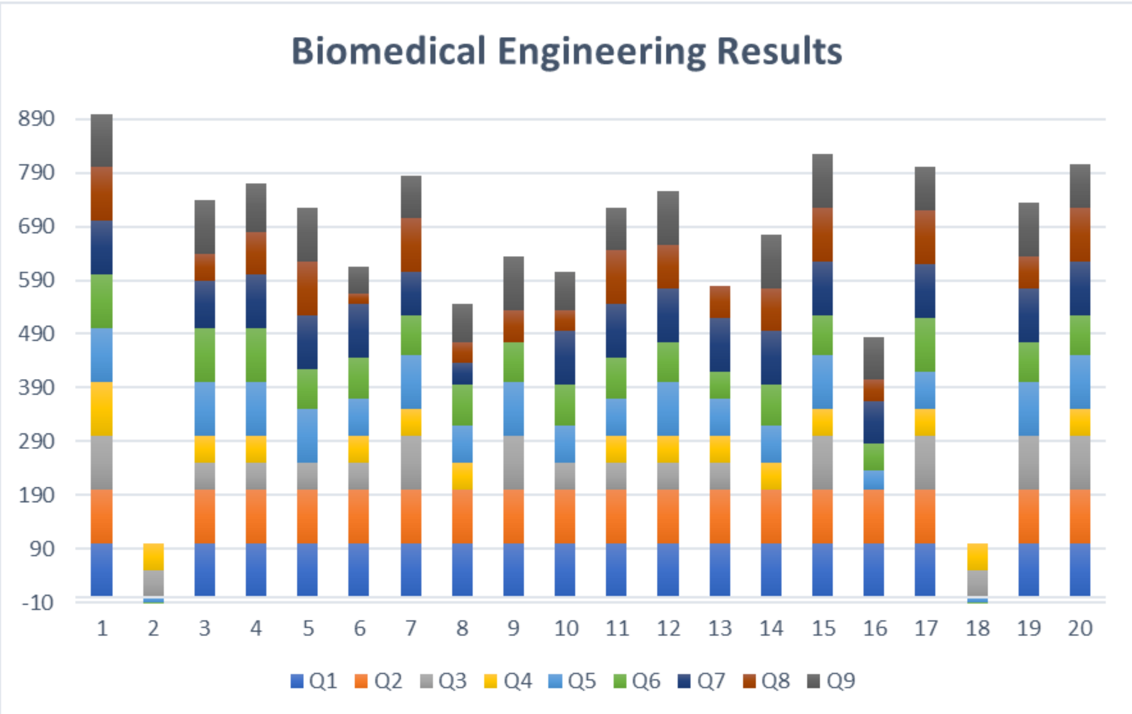

Figure 4. Biomedical Engineering Results for all the students (20).

The average of the responses of Biomedical students to the 9 questions is 90% (Q1, Q2), 60% (Q3), 40% (Q4), 84.7% (Q5), 77.8% (Q6), 83.7% (Q7), 72.8% (Q8) and 82.2% (Q9). Thus, the questions that got the best results were Q1 and Q2 and the question that got the worst results were Q4 with a negative value (40%), In the remaining questions the average is above 72%.

The results obtained by the 27 students of Electromechanical Engineering can be seen in Figure 5.

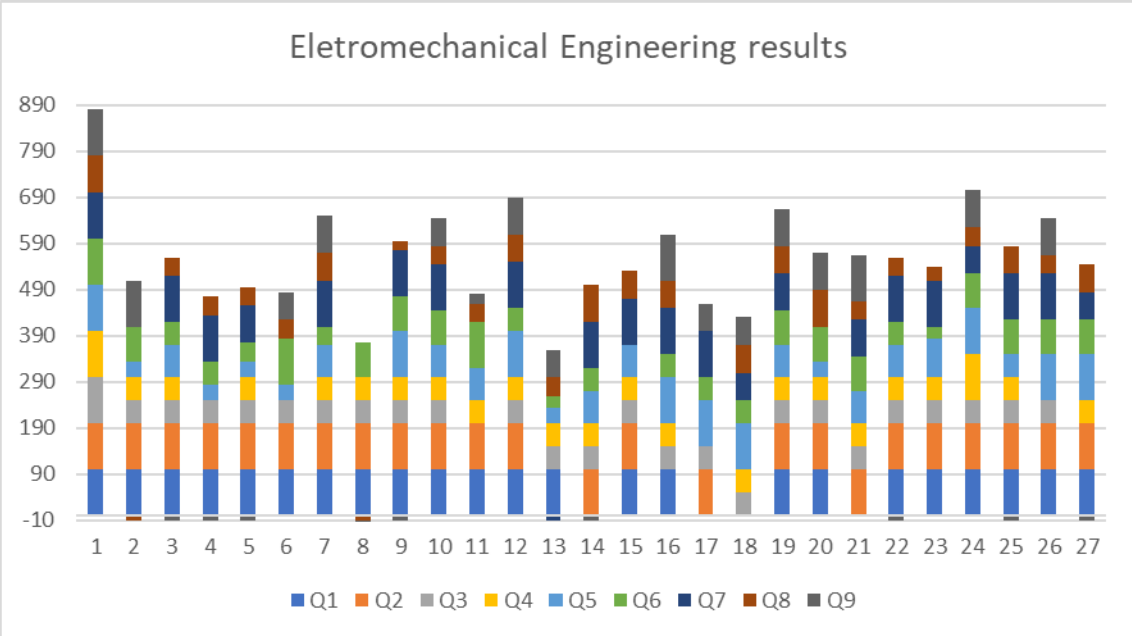

Figure 5. Eletromechanical Engineering Results for all the students (27).

In the course of electromechanics, the average answers were: 85.2% (Q1), 88.9% (Q2), 48.1% (Q3), 46.3% (Q4), 69.4% (Q5), 61.3% (Q6), 73.8% (Q7), 48.4% (Q8) and 66.7% (Q9). In this course, there are three questions with an average below 50% (Q3, Q4 and Q8), with Q4 being the question that had the worst students performance (46.3%). The best students performance was on question Q2 with almost 89%. In the remaining questions, students got an average of more than 61%.

Figure 6 shows all the results of the 79 Mechanical Engineering students.

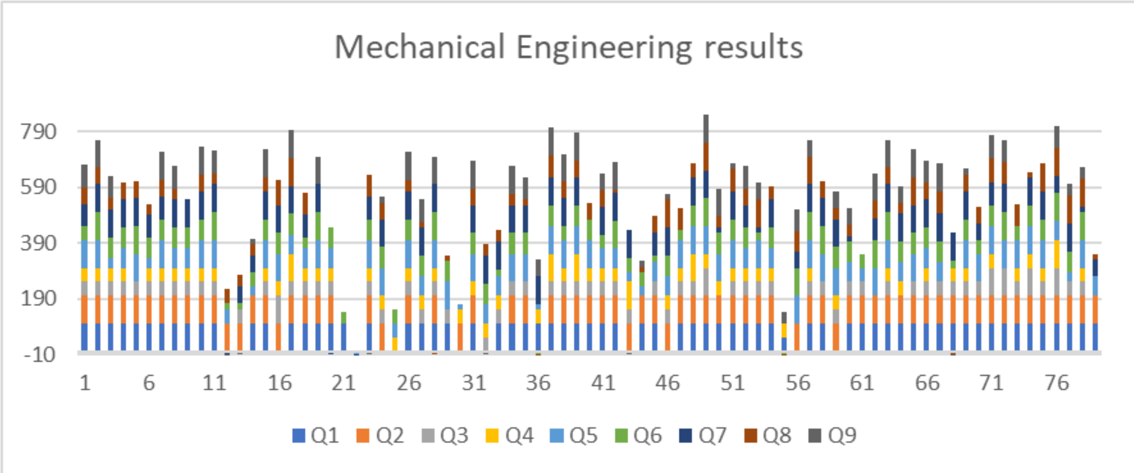

Figure 6. Mechanical Engineering Results for all the students (79).

In Mechanical engineering, the average answers were: 84.2% (Q1), 88.6% (Q2), 44.3% (Q3), 39.9% (Q4), 77.8% (Q5), 68.4% (Q6), 77.1% (Q7), 53.8% (Q8), 50.3% (Q9). Questions Q3 and Q4 have an average of less than 50%, with Q4 being the question with the worst result. On the other hand, the question with the best result is question Q2 with around 87%.

Question 2 (Q2) was the one with the best results in all courses (Figure 7), 18 students out of 20 from Biomedical Engineering, 24 from 27 from Electromechanics and 70 from 79 from Mechanics answered correctly. The remaining students (14 students) answered wrongly, getting a score of zero.

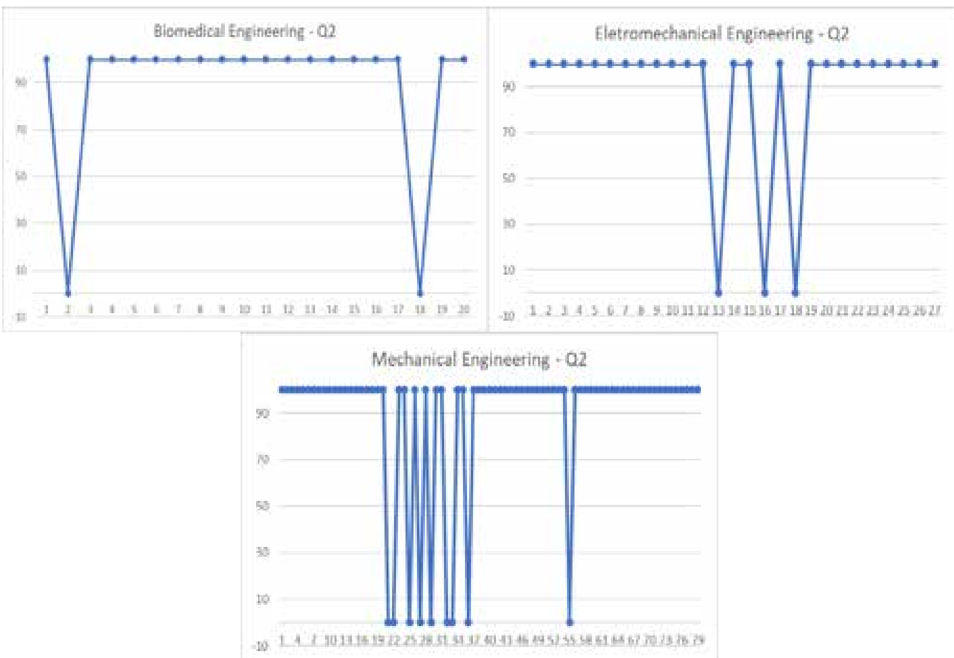

Figure 7. Results obtained in question Q2 for the 3 engineering courses.

The question with the best results in the 3 courses is question 4 (Figure 8 to 10). In the case of Biomedical Engineering (Figure 8, top), only 1 student answered correctly (100%), 14 students answered partially correct (50%) and 5 students answered wrong (0%). In the case of Electromechanical Engineering (Figure 9, top), only 2 students answered correctly, 21 students answered partially correct (50%) and 4 students answered wrong (0%). Finally, in Mechanical Engineering (Figure 10, top), 6 students obtained 100%, 51 students obtained 50% and 22 students obtained 0%.

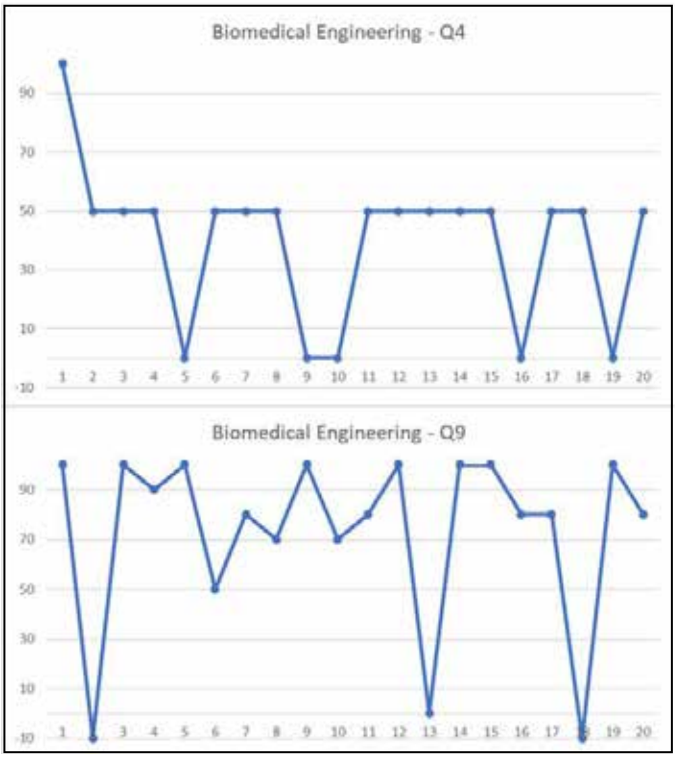

Figure 8. Results obtained in question Q2 (top) and Q9 (bottom) for the Biomedical Engineering

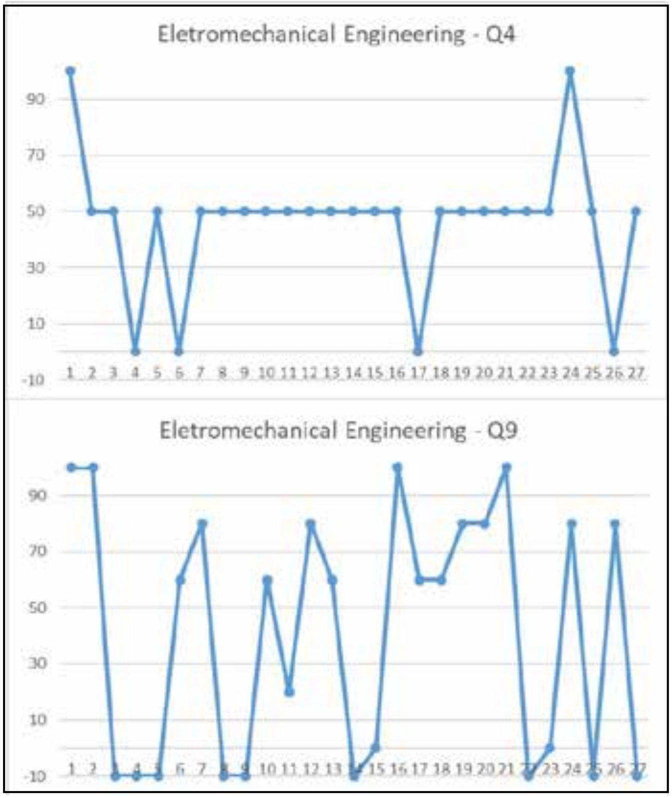

Figure 9. Results obtained in question Q2 (top) and Q9 (bottom) for the Eletromechanical Engineering

Regarding question 9 (Q9), the one that involves all learning objectives as well as all mathematical competences (with the exception of competence 9, which was not evaluated in this test) the results are represented in Figures 8 to 10 (bottom). The results obtained in the different engineering are: 100% (8 Biomedical and Electromechanical students, 20 Mechanical students), 50% (1 Biomedical student, none Electromechanical and Mechanical students), 0% (1 Biomedical student, 2 Electromechanical students, 24 Mechanical students), no answer (2 Biomedical students, 9 Eletromechanical students and 7 Mechanical students), the remaining percentages obtained in the answers, were quite diversified.

Since the questions asked to the students from the different courses were the same, it is observed that the results are a little different between the 3 engineering courses. Biomedical Engineering has the best average in all answers (75.7%) after, the Electromechanical Engineering (65.4%) and lastly Mechanical Engineering (64.9%).

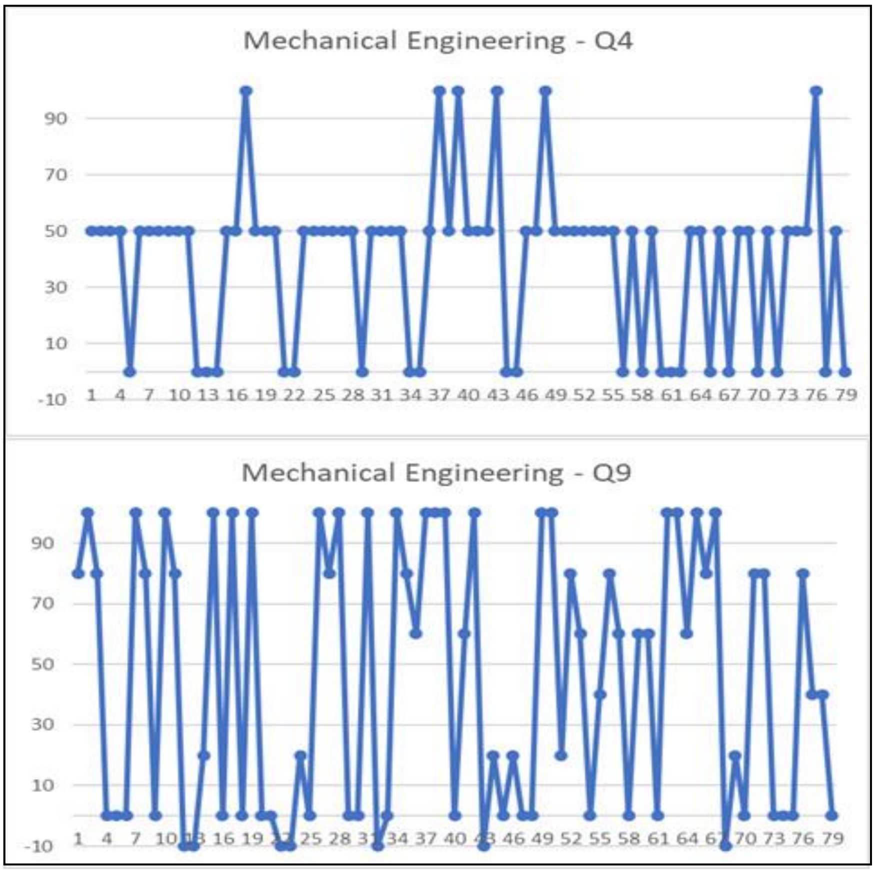

Figure 10. Results obtained in question Q2 (top) and Q9 (bottom) for the Mechanical Engineering.

Conclusions

From the results obtained, it was found that students from the three courses acquired the learning objectives and mathematical skills of the contents taught. Despite the results obtained in questions 3 and 4, below 50%, by the students of the Electromechanical and Mechanical Engineering disciplines, in general all the students obtained positive marks. However, there are still some students who have negative results in this content and therefore need greater support from the teacher, either in terms of learning objectives or in terms of the mathematical skills involved. Perhaps, through an analysis like this done in timely manner, it will be possible to identify the students, and in which mathematical competences and learning objectives need more attention. In order to get them to overcome their specific difficulties the teacher may encourage students to use office hours, support classes or classes zooming in on doubts can be a way to reduce the negative cases that have been identified.

On the other hand, the construction of evaluation material that assesses all the contents to be taught learning objectives, and all the mathematical competences that an engineer must possess, is the main objective of the RULES_MATH project. The idea that mathematics taught to future engineers should be based on learning objectives and mathematical competences should be followed by teachers, both when teaching students and when evaluating, is the RULES_MATH objective. Therefore, it is important to continue using these criteria with the same or similar exercises as those presented in this paper. It is necessary that teachers adjust their teaching and their assessment to these criteria, it is necessary to publicize the RULES_MATH group and its objectives to all those who teach mathematics to engineers.

The technique developed can be easily applied by any teacher in any area of mathematics to assess the quality of your questions and the quality of your students' responses in relation to the learning objectives and mathematical competencies involved. The model developed allows to prepare questions to be asked by the teacher to his students that ensure that all learning objectives and mathematical competences are evaluated. The technique presented here can and should be used by teachers in preparing their students' assessment exercises. On the other hand, it allows to verify if the student has acquired all the learning and competences involved. The paper does not examine the difficulty of the questions being asked or the variety and frequency of the type of question being asked.

However, it is essential to develop processes that help understand the difficulty of engineering students in mathematics. In which learning objectives students have more difficulties and what mathematical competencies are more difficult for future engineers to acquire. We will continue applying this model in the teaching and evaluation of engineering students and we encourage other teachers to use it too.

Acknowledgement

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References

Alpers, B. et al, (2013) "A framework for mathematics curricula in engineering education." SEFI, 2013. Available online at: http://sefi.htw-aalen.de/

Boesen, J., Lithner, J., Palm, T. (2018) "Assessing mathematical competencies: an analysis of Swedish national mathematics tests." *Scandinavian Journal of Educational Research*, 62:1, 109-124, DOI: 10.1080/00313831.2016.1212256

Caridade C.M.R., Rasteiro D.M.L.D. (2019). "Evaluate Mathematical Competencies in Engineering Using Video-Lessons." *In: Martínez Álvarez F., Troncoso Lora A., Sáez Muñoz J., Quintián H., Corchado E. (eds) International Joint Conference: 12th International Conference on Computational Intelligence in Security for Information Systems (CISIS 2019) and 10th International Conference on European Transnational Education (ICEUTE 2019). CISIS 2019, ICEUTE 2019. Advances in Intelligent Systems and Computing*, vol 951. Springer, Cham.

Caridade, C.M.R., Rasteiro, D.M.L.D., Richtarikova, D. (2020). "Mathematics for engineers: assessing knowledge and competencies." *In Conference Proceedings Aplimat Conference*. 4-6 Feb 2020. Bratislava.

Niss, M., (2003) "Mathematical Competencies and the Learning of Mathematics: The Danish KOM Project." *In Proceedings of the 3rd Mediterranean*.

Niss, M., Højaard, T. (2011) "Competencies and Mathematical Learning. Ideas and inspiration for the development of mathematics teaching and learning in Denmark." *English Edition, Roskilde University*.

Niss, M., Bruder, R., Planas, N., Turner, R., Violla-Ochoa, J.A. (2017) "Conceptualisation of the Role of Competencies, Knowing and Knowledge in Mathematics Education Research." *In: Kaiser G. (eds) Proceedings of the 13th International Congress on Mathematical Education. ICME-13 Monographs.* Springer, Cham.

Ross, T. (2010) „Exploring mathematical competencies." 24:5. Available at: https://research.acer.edu.au/resdev/vol24/iss24/5

RULES_MATH project. https://rules-math.com/; https://rulesmath.wixsite.com/euproject

Mathematical Reasoning in Engineering Statics

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Abstract

Mathematical reasoning has been recognised as an important part of mathematical competence. In order to find out which aspects of this competency are important in a certain study course, the main application subjects of this study course should be investigated regarding the kind of (mathematical) reasoning that is necessary to follow theory development and solve problems. In this contribution a part of a widely used statics textbook is analysed regarding definitions, axioms, theorems and reasoning, and discrepancies with what is accepted practice in mathematics are identified. Potential consequences for the mathematical education of engineers are discussed.

Introduction

Mathematical reasoning is one of eight mathematical competencies that are stated in the curriculum document of SEFI's Maths Working Group (Alpers et al. 2013). This was elaborated in more detail for a practice-oriented study course in mechanical engineering in (Alpers 2012) based on the experience of the author with the kind of reasoning needed in such a course. Such experience should be substantiated by systematic investigation into the reasoning to be found in application subjects, particularly in theory development in such subjects. In this contribution, I look at engineering statics, one of the first and most basic modules of a study course in mechanical engineering. I analyse a few basic chapters of a textbook that is widely used in Germany (Gross et al. 2016, 13. Edition) and that has also been translated into English (Gross et al. 2013).

I will first elaborate on the concept of "mathematical reasoning" as it is discussed in literature in order to make available some categories that can be used for analysis. Moreover, I will present a few results on differences between mathematical practices in mathematics and other subjects. Then follows a presentation of the method of investigation and the main results. The contribution closes with a discussion of potential consequences for the mathematical education of engineers.

Theoretical background and previous work

Brunner (2014) found out that the terms "arguing", "providing reasons/reasoning", and "proving" are used with different meanings in literature (see also Reid & Knipping 2010 as an English reference). She suggests to use "reasoning" as most comprehensive term. On a continuous scale of reasoning she positions the sub-types "everyday arguing", "arguing with mathematical means", "logical arguing with mathematical means" and "formal-deductive proving" (p. 31). The sub-type "arguing with mathematical means" includes the inspection of examples without logical deduction which provides plausibility arguments. "Logical arguing with mathematical means" includes logical reasoning without a formal context (terms, symbols), like reasoning in 3D space using terms and properties "known" from experience like points, lines, parallels that can be drawn. Finally, "formal-deductive proving" is what could be called mathematical reasoning in a narrower sense as accepted in mathematics as a science. It is my understanding that within this spectrum the mathematical reasoning competency as discussed in (Alpers et al. 2013) comprises the latter two sub-types.

Hochmuth & Schreiber (2016) use the "Anthropological Theory of Didactics" (ATD) to investigate different so-called praxeologies in textbooks on signal and system theory (SST, 4th semester) and on engineering mathematics. They state that the basic terms in SST are introduced as mathematical objects carrying application meaning, so there is no separation between the „real world“ and its mathematisation as proclaimed in the construct of "modelling cycle". When the Dirac impulse (mathematically: δ-distribution) is introduced and properties are investigated, plausibility arguments are provided like the approximation by functions and the properties of corresponding integrals. But a formal argumentation using the definition of distributions is missing. The textbook authors refer to mathematics but also to real practice which justifies the usage of the Dirac impulse since it provides useful results. In the end, students have to „neglect specific elements of discourses in higher mathematics" (p. 558, translation: B.A.). Using Brunner's spectrum the reasoning can be positioned in the area of „arguing with mathematical means“ providing plausibility using graphical illustration and usage of mathematical statements without consideration of necessary prerequisites.

Uhden et al. (2012) investigate the relationship between mathematics and theory development in physics. They state that the latter cannot be regarded as separate from mathematics, being mathematised only later. Beside an instrumental role of mathematics when doing computations, mathematics also has a structural role since physical principles are usually developed in a „joint physical-mathematical model“ (p. 491). Model development is elaborated by further mathematisation and interpretation of results of mathematical argumentations. Moreover, there is also an area of purely qualitative physics.

Methods and results

Engineering statics is a basic and important subject in any study course on mechanical engineering. In the textbook by Gross et al. (2016) on engineering statics the first chapters on „Basic Concepts“, „Forces with a common point of application", "General Systems of Forces, Equilibrium of a Rigid Body", as well as on "Center of Gravity, Center of Mass, Centroids" have been investigated guided by the following questions: How are new terms and concepts introduced (formal definition, mathematical representation, degree of exactness)? Are assumed properties (axioms) stated explicitly? Is the difference between axioms and theorems made explicit? Where in the reasoning continuum suggested by Brunner can the reasoning applied in theory development be placed. Do the authors use mechanical and/or mathematical arguments? How are mathematical results used (explicit reference or implicit assumption)? In the textbook, there is also a separate mathematical annex containing sections on "Elements of vector algebra" and "Linear systems of equations".

The most basic concept used in statics is that of a force. Force is defined as "a physical quantity that can be brought into equilibrium with gravity" (p. 7). For further elaboration of this term geometrical properties like magnitude (length), line of action (direction) and point of application are introduced and a reference to the geometrically-illustrative mathematical definition of a free vector (quantity determined by absolute value and direction) is given. Then the authors use the formal representation of a vector as a triple of real numbers based on a given coordinate system. As stated by Uhden et al. (2012) for physics in general, in the development of central terms mathematics and application (statics) are inseparably interwoven. Yet, the term force is not directly introduced as a mathematical object as was observed by (Hochmuth & Schreiber 2016) in books on systems and signals. The latter happens when the authors introduce forces distributed over lines, areas or volumes which are defined as functions (including units).

In the introduction to the book, the authors state: „Mechanics is based on only a few laws of nature which have an axiomatic character. These are statements based on numerous observations and regarded as being known from experience. The conclusions drawn from these laws are also confirmed by experience. Mechanical quantities such as velocity, mass, force, momentum or energy describing the mechanical properties of a system are connected within these axioms and within the resulting theorems.“ (p.1). So, the authors distinguish between axioms and conclusions/theorems. But in the subsequent chapters this difference is blurred. Regarding the axiom „parallelogram law of forces“ one can recognize a remarkable difference between the German and English edition. In the German edition it says (nearly word-by-word translation by B.A.): „This fact drawn from experience is formulated in the theorem on the parallelogram of forces. The theorem states … We can also formulate this axiom as follows …“ (Gross et al, 2016, p.21, see also p. 28). The terms “fact drawn from experience“, „theorem“ and „axiom“ seem to be used synonymously. In the English edition, the authors (presumably the additional native speaking author) state: “This postulate is an axiom. It is known as the parallelogram law of forces. …The axiom may be expressed in the following way …” (p. 21). In both editions, there is no formal or graphical separation between axioms and theorems. Important statements are depicted with blue background independent of their logical character. The reader has to compile the set of axioms him/herself (forces can be moved along their line of action, actio=reactio, forces are in equilibrium when they are on the same line of action, have the same magnitude and different directions). Theorems are usually formulated as a result of former considerations and derivations. In the English edition, only at one place a statement is formulated first and then proved (p. 64: “In order to prove this statement …”). The theorem states that in 2D-problems one can set up equilibrium equations for moments using 3 points arbitrarily as long as they are not collinear. In the German edition which contains more theory, there is an additional theorem stated explicitly saying that there is one special point such that all forces and moments can be equivalently substituted by one force and one collinear moment vector (Gross et al. 2013, p. 84).

In the argumentations presented in the derivations of theorems one can distinguish between purely mechanical, mixed mechanical-mathematical and purely mathematical arguments. The purely mechanical arguments apply the axioms of statics in 3D-space using arrows as illustrative geometrical representations and making also implicit use of geometrical axioms like Euclid’s axiom on the existence and uniqueness of a line which contains a given point P and is parallel to a given line. Such kinds of argumentations could be characterised as “logical arguing with mechanical means” comparable with the category “logical arguing with mathematical means” within Brunner’s (2014) spectrum. As an example, we consider the argumentation for substituting a group of 2D-forces by one force and a moment (p. 60): A force can be moved to a specific point by attaching two forces with opposite directions and the same magnitude as the original force (this is also called “equilibrium group”, see figure 1 first row). Then the original force and one of the “new” forces can be combined to a moment, and one force remains. This way, all forces can be moved to one common point and then substituted by one force using the parallelogram law. All moments can also be replaced by one moment (see figure 1, second row). This reasoning contains typical mechanical arguments.

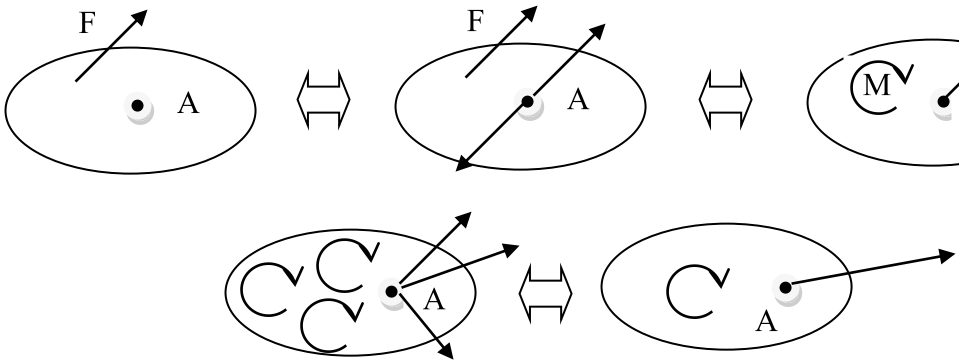
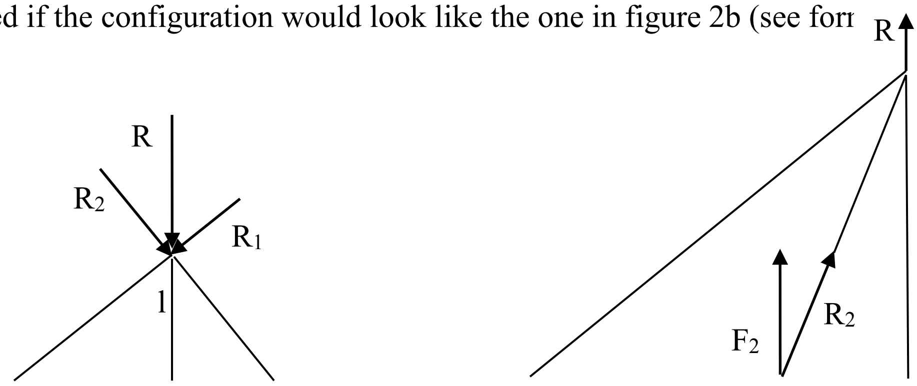


Figure 1

Mechanical arguments could also be stated in a formal mathematical way when the term „bound vector“ (i.e. vector bound to a line of action instead of a „free vector“) is formalised enabling formal proofs. This has been shown in the book on mathematics for engineers by Meyberg & Vachenauer (2001) where there is a specific chapter on the topic which also contains a proof of the theorem on the existence of a special point such that all forces and moments can be equivalently substituted by one force and one collinear moment vector. But it seems to be obvious that for an average engineering student the formal reasoning is much harder to understand. Hence, Meyberg & Vachenauer (2001) mark the specific chapter as optional.

Mathematical arguments (provided together with mechanical ones or “stand alone”) are frequently of geometric nature and can hence be located in Brunner’s category of „logical arguing with mathematical means”. Often, the reasoning is related to a specific sketch drawn in the book. There are no considerations whether the arguments are still valid in a different configuration, i.e. the property of „exemplarity“ of the argumentation is not clarified. As an example we consider two parallel forces F_1 and F_2 which are combined into one (p. 53, see figure 2 below). Two forces K and –K are added (equilibrium group). Then, the parallelogram law is applied to F_1 and K as well as to F_2 and –K. If F_2 is not equal to $-F_1$, the lines of action of the resulting forces intersect and one can combine them into one force. So far, the argument does not depend on the geometric configuration depicted in figure 2a (containing the distances a_1, a_2, l, h). But the subsequent computation of distances depends on the configuration and have to be adapted if the configuration would look like the one in figure 2b (see for[illegible] for h).



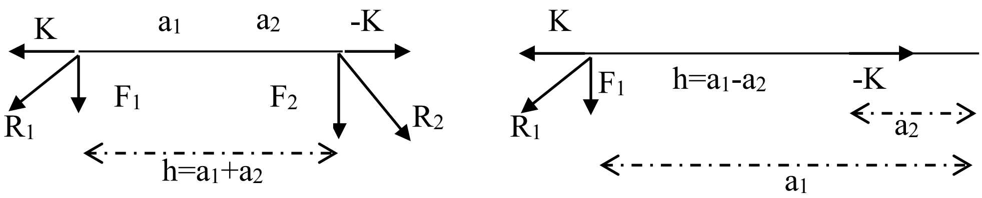


Figure 2 a and b

Additional formal mathematical reasoning can be found in computations related to solving systems of equations (and considerations on solvability) as well as computations in vector algebra. It is remarkable that no difference is made between implication and equivalence when reformulating problems. Hence, from a logical point of view arguments are sometimes not correct. For example, when equations are modified by squaring both sides there is no check whether the solutions obtained in the end really solve the initial equation. In the statement on the existence of a special point for substituting a system of forces and moments by one force and one moment in the German edition (p. 60, see above) the authors start with assuming that such a point exists and then determine the point without giving any consideration on the reverse direction.

In the chapter on "Center of Gravity" etc., one can find argumentations using differentials dx, dA, dV as is well known from physics. In Alpers (2017), differences in using differentials in mathematics and statics have already been investigated. There, it is recommended to explain the differential approach to students as a "shortcut" where all limits are assumed to exist. Interpreted in terms of mathematical reasoning one could state that the arguments are exemplary for all cases where the limits exist but this is not made explicit in the statics textbook.

To sum up, our investigation shows that within the core area of mathematical argumentation ("logical arguing with mathematical means" and "formal-deductive proving") there are remarkable differences between what is done and postulated in mathematics education and what can be found in the statics book.

Potential consequences for education

Since the statics book is quite widespread in Germany, it seems to be clear that the idea that students acquire mathematical reasoning in mathematics education and then use it in application subjects does not reflect adequately the situation of mathematical argumentation in such subjects. Since we just investigated one book it might be the case that in other books or in other application subjects the situation turns out to be different but our investigation rather gives rise to scepticism. Considering Brunner's spectrum of mathematical reasoning, there is no need to just allow a formal-deductive approach. Logical reasoning with mathematical means rather seems to be the appropriate kind of reasoning to strive for. But even then, there should be a clear distinction between axioms and theorems as well as between implication and equivalence, and students should be aware of the problem of exemplarity when they argue using a geometrical sketch. We cannot expect a student to take the emphasis serious that is laid on these issues in mathematics education when they play no role in application subjects. This rather fosters views where mathematics is considered as isolated from the real world.

It is the intention of this contribution to stimulate a discussion between mathematics educators and lecturers in application subjects on the issue of mathematical argumentation. Only if there is a coherent approach in both areas can we expect students to recognise mathematical argumentation as important and relevant and hence to be willing to spend the necessary effort for acquiring the respective competency.

References

Alpers, B. (2012). The mathematical reasoning competency for a practice-oriented study course in mechanical engineering. Proc. 40th SEFI Ann. Conf., Brussels: SEFI.

Alpers, B. et al. (2013). A framework for mathematics curricula in engineering education. Brussels: SEFI.

Alpers, B. (2017). Differences between the usage of mathematical concepts in engineering statics and engineering mathematics education. R. Göller et al. (Eds.) Didactics of Mathematics in Higher Education as a Scientific Discipline – Conference Proceedings. Khdm-Report 17-05 (pp.137-141). Kassel: Universität Kassel.

Brunner, E. (2014). Mathematisches Argumentieren, Begründen und Beweisen. Grundlagen, Befunde und Konzepte. D: Springer Spektrum.

Gross, D. et al. (2013). Engineering Mechanics 1. Statics. 2. Ed., Dordrecht: Springer.

Gross, D. et al. (2016). Technische Mechanik 1. Statik. 13. Auflage, Berlin-Heidelberg: Springer.

Hochmuth, R., Schreiber, S. (2016). Überlegungen zur Konzeptualisierung mathematischer Kompetenzen im fortgeschrittenen Ingenieurwissenschaftsstudium am Beispiel der Signaltheorie. Hoppenbrock, A. et al. (Eds.). Lehren und Lernen von Mathematik in der Studieneingangsphase, pp. 549-566, Wiesbaden: Springer.

Meyberg, K., Vachenauer, P. (2001). Höhere Mathematik I, 6. Ed., Berlin: Springer.

Reid, D.A., Knipping, C. (2010). Proof in Mathematics Education: Research, Learning and Teaching. Rotterdam: Sense Publ.

Uhden, O. et al. (2012). Modelling Mathematical Reasoning in Physics Education. Science and Education 21, pp. 485-506.

Variations of engineering students' attitude towards mathematics across gender and age: A MIMIC model approach

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Abstract

Researchers across the world have focussed attention on the relationship between students' attitude towards mathematics and learning outcomes in the subject. There seems to be a scarcity of careful studies on the psychometric properties of attitude towards mathematics scales. The available studies are limited to establishing factor structures of the scales and computing reliability indices. The present study is a follow-up research on the validity of an attitude towards mathematics questionnaire that has been in use for over two decades in Norway. The purpose of the study is twofold. First, to confirm the factor structure of the questionnaire in an independent sample from the initial study. Second, to investigate the variations of the questionnaire across gender and age of first-year engineering students. Three research questions are raised and addressed. The data are generated using a student attitude toward mathematics questionnaire and analysed with a Multiple Indicators, Multiple Causes (MIMIC) structural equation modeling technique. The results confirm a single factor structure for the five-item questionnaire. As an improvement on the previous study that recommends two error covariances between some items of the questionnaire, only one error covariance is found to be sufficient to achieve the single factor structure of the questionnaire. Further, evidence from the present study shows that the attitude toward mathematics questionnaire is neither gender-biased nor age-biased. As such, the questionnaire can be used to measure engineering students' attitude towards mathematics regardless of gender and age of the students.

Introduction

Research on affects in mathematics education has generated heated debates among mathematics educators. The conceptualisations of, and the theoretical foundations for constructs such as attitude, anxiety, beliefs, motivation, emotions, and values as they relate to mathematics learning have been argued extensively (Hannula, 2012, Zan et al., 2006, McLeod, 1992). Students' attitude towards mathematics, according to McLeod (1992), is conceptualised as "affective responses that involve positive or negative feelings of moderate intensity and reasonable stability'' (p. 581). From this viewpoint, students' attitude towards mathematics is relatively more stable than students' emotion but less stable than students' belief in mathematics. Other researchers (e.g., Di Martino and Zan, 2009, Yavuz Mumcu and Cansız Aktaş, 2020) have conceptualised students' attitude towards mathematics to comprise cognitive (e.g., knowledge and beliefs), affective (liking-disliking emotions), and behavioural (e.g., tendency to approach or refrain from mathematics activities) components. Attitude towards mathematics may be defined, tentatively, as students' like/dislike and perceived usefulness of mathematics. Despite the lack of coherence in the theoretical foundations of the students' attitude towards mathematics, there seems to be an accumulation of evidence that establishes a strong contribution of the construct to students' learning outcomes in mathematics.

Previous studies show that students' attitude towards mathematics has non-trivial effects on students' performance in mathematics. Lipnevich et al. (2016) reported a two-sample study in which the contributions of university students' attitude towards mathematics on self-reported mathematics grades are investigated. Their findings show that students' attitude towards mathematics predicts students' mathematics performance better than some cognitive and personality constructs e.g., reasoning ability, extraversion, and openness. In another study involving 240 pupils in the United States, Chen et al. (2018) found that positive attitude towards mathematics has a substantial effect on students' achievement in mathematics. Some other researchers have investigated the reciprocal effects of attitude towards mathematics and students' performance in mathematics and found that there is non-trivial reciprocal effects between the constructs (Kiwanuka et al., 2020). That is, previous attitude affects subsequent performance while current performance affects subsequent attitude towards mathematics.

Considering the crucial role of students' attitude towards mathematics on performance, one may be interested to carefully check the measurement of the construct. Admittedly, there have been some previous studies with this intention. However, some of the studies are exploratory/confirmatory of factor structure of the instrument or estimation of factor scores (e.g., Ayob and Yassin, 2017). Some other studies are based on developing a new instrument or preparing a short form of an existing one (Yavuz Mumcu and Cansız Aktaş, 2020, Primi et al., 2020). Meanwhile, there is a scarcity of studies on how the construct varies across gender and age. The present study is a follow-up research on the validity of an attitude towards mathematics questionnaire that is used every two years to measure Norwegian students' attitude towards mathematics. The purpose of the present study is twofold. First, to confirm the factor structure of the questionnaire in an independent sample from the initial study. Second, to investigate the variations of the questionnaire across gender and age of first-year engineering undergraduate students. In specific terms, the following questions are raised: Do the five items of attitude towards mathematics questionnaire measure a single factor? Does the attitude towards mathematics vary across students' gender? Does the attitude towards mathematics vary across students' age? The author believes that empirical evidence to support answers to these questions will inform decisions on the use of this questionnaire for subsequent gender and age comparisons across universities in Norway.

Methods

Participants and measure

The sample of the study comprised 238 (189 males) first-year engineering students enrolled in engineering programmes at a Norwegian university. Their age is distributed into the following intervals: 17-20 years (109), 21-25 years (98), 26-35 years (27), and over 36 years (4). Majority of the students are Norwegians, and the language of instruction is Norwegian. The instrument used in the present study, attitude towards mathematics questionnaire (AtMQ), is part of a national mathematics test that is administered every two years to first-year degree-seeking students in Norway. It is a five-item questionnaire that is designed to measure a single construct of students' attitude towards mathematics on a four-point Likert scale format from *strongly disagree* to *strongly agree*. The students are to rate their agreement to a sample-item statement like, *I work with math because I like it*, in a response to the leading question of *what are your attitudes toward mathematics*? Previous validation research on AtMQ has reported a good discriminant validity, reliability (ordinal coefficient alpha of .78), and construct validity of the instrument (Zakariya et al., 2020).

Data collection

The data used for the present study are generated using online version of AtMQ that was administered to first-year engineering students via a SurveyXact link. The data on students' gender are coded as 1 for males and 0 for females. The data on age groups are collapsed such that students between 17-20 years have a coded of 1 and those above 20 years have a code of 2 to indicate young and old students, respectively. As such, there are 109 young students and 129 old students. The age-group recoding is necessary to facilitate the use of Multiple Indicators, Multiple Causes (MIMIC) modelling. A preliminary analysis show that the data contain neither excess Kurtosis nor excess Skewness. However, the data are not normally distributed as revealed by the significant values of both Kolmogorov-Smirnov's and Shapiro-Wilk's tests for all items.

Data analysis

The data are analysed with some techniques of structural equation modelling. The weighted least square mean and variance adjusted (WLSMV) estimator was used to cater for the lack of normal distribution of the data (Suh, 2015, Zakariya, 2020). A one factor five-item measurement model of AtMQ was evaluated against the generated data. The model global fit was assessed using a combination of criteria. The variations of attitude towards mathematics, at the construct level, across gender and age are investigated with the MIMIC modelling. MIMIC modelling is a structural equation modelling technique that is suited for investigating factor invariance across group memberships (covariates) by regressing the latent factor onto the covariate (Muthén, 1989). The idea of the MIMIC modelling is similar to the mean comparison tests (t-test or analysis of variance) in classical test theory. However, mean comparisons in MIMIC modelling take place at latent construct level rather than in observed scores as it is usually the case in classical test theory. The mechanism of MIMIC modelling involves an evaluation of measurement model using group membership as a covariate. In the present study, both gender and age group of students are used as covariates to investigate their effect on the attitude towards mathematics. A significant effect of each of these covariates will suggest that AtMQ does not measure attitude towards mathematics, in a similar proportion, across each of the students' gender types (female and male) and the membership of the age groups (old or young). In contrast, non-significate effects of the covariates will confirm the measurement invariance of AtMQ across the students' gender types and age groups.

Results and discussion

Research question one

The first set of results in the present study concerns an investigation into whether the five-item AtMQ measure a single construct. Three different measurement models of the AtMQ were evaluated using confirmatory factor analyses. The first model (Model 1) was a single-factor measurement model of AtMQ without any error covariance. The second model (Model 2) had one error covariance between item 2 and item 4, while the third model (Model 3) had two error covariances as recommended by (Zakariya et al., 2020). Table 1 shows the goodness of fit statistics of the confirmatory factor analysis results.

Table 1. Goodness of fit statistics of three models of one-factor AtMQ

	Model 1	Model 2	Model 3*
χ^2-value	75.367	6.455	3.836
df	5	4	3

p-value	<.001	.168	.280
CFI	0.872	.996	.998
TLI	0.744	.989	.995
RMSEA (90% CI)	0.243 (.196 – .293)	.051 (<.001 – .120)	.034 (<.001 – .120)
Probability RMSEA ≤ .05	<.001	.411	.512
SRMR value	0.131	.028	.020

*The latent variable covariance matrix (psi) is not positive definite

The results presented in Table 1 show some goodness of fit statistics of three models of one-factor AtMQ. The results under Model 1 show an unacceptable model fit of the generated data. This is because the chi-square value is significant, and all the fit indices are not within the acceptable criteria of a good model. As such, one can infer that a one-factor measurement model of AtMQ without correlated errors is not consistent with the generated data. Model 2 is an improvement on Model 1 that allows one error covariance between item 2 and item 4. This improvement on Model 1 is informed by previous studies (e.g., Zakariya et al., 2020) and the statistical arguments (modification indices). The results as presented under Model 2 in Table 2 show an excellent model fit of the data. The chi-square value is not significant and all the goodness fit indices are within the acceptable criteria of an excellent model (Hu and Bentler, 1999). As such, one can infer that a one-factor AtMQ model with one error covariance between item 2 and item 4 is consistent with the generated data. Following the recommendation by Zakariya et al. (2020), the author investigated Model 3 with two error covariances. Despite the excellent model fit as shown by the fit statistics, the lack of positive definiteness of the covariance matrix renders the model unacceptable (Byrne, 2012). Therefore, Model 2 is the best measurement model of AtMQ that is consistent with the generated data.

Research questions two and three

The next set of results concerns the questions on whether AtMQ measures attitude towards mathematics, in similar proportions, across male and female students, and across old and young students. Using the MIMIC modelling, the attitude towards mathematics construct was regressed on gender and age-group. The results of these analyses are presented in Table 2 with Model 4 for gender covariate and Model 5 for age-group covariate.

Table 2. Goodness of fit statistics for measurement invariance of AtMQ across gender and age groups

	Model 4 (Gender)	Model 5 (Age-Group)
χ^2-value	14.665	16.711
df	8	8
p-value	.066	.0333
CFI	.989	.985
TLI	.979	.971
RMSEA (90% CI)	.059 (< .001 – .106)	.068 (.018 – .113)
Probability RMSEA ≤ .05	.327	.226
SRMR value	.052	.044
Effect	.010	.008
p-value	.903	.922

The results in Table 2 (Model 4) suggest that an excellent model fit of the measurement model should be retained after regressing the latent construct on students' gender type. The retainment of the excellent model is inferred from the non-significant chi-square value and goodness of fit statistics that are within the recommended criteria for an excellent model fit. Interestingly, the results show that the effect of gender on the attitude towards mathematics construct is not significant. That is, the AtMQ measures what is purported to measure regardless of whether the respondents are males or females. This is a crucial finding in the present study as it reinforces confidence in the questionnaire for subsequent use in comparing male and female attitude towards mathematics.

The results in Table 2 (Model 5) also suggest that an excellent model fit of the AtMQ measurement model should be retained after regressing the latent construct on students' age-group. The retainment of the excellent model is inferred from the goodness of fit statistics that are within the recommended criteria for an excellent model fit. One may observe that the chi-square value is significant ($p < .05$) which could be a failure of the measurement model to exhibit a global fit of the generated data. However, in such an instance, some researchers (e.g., Kline, 2016) have recommended that if the ratio of the chi-square value to the degree of freedom (df) is less than 3, then the model exhibits an appropriate global fit. More so, the RMSEA value is a little above .06 but it poses no problem to the model fit since the 90% confidence interval includes .06 (Zakariya, 2021). Interestingly, the results show that the effect of age-group on the attitude towards mathematics construct is not significant. That is, the AtMQ measures what is purported to measure regardless of whether the respondents are old or young. This is a crucial finding in the present study as it reinforces confidence in the questionnaire for subsequent use in comparing attitude towards mathematics across students of different age groups.

Conclusion

Students' attitude towards mathematics is an important construct that influences students' performance in mathematics even if there is no consensus among researchers on the conceptualisation of the construct. On this premise, one may argue that a careful development and measurement of the attitude towards mathematics is non-negotiable. Thus, the present study was motivated with this intention. As a follow-up study, the present study investigated the construct validity and gender/age invariance of an attitude towards mathematics questionnaire. The most crucial finding of the present study is the invariance of attitude toward mathematics questionnaire across students' gender and age groups. The implication of this finding goes to researchers, in general, and the Norwegian National Mathematical Council to continue the use of AtMQ for measuring engineering students' attitude towards mathematics regardless of gender and age of the students.

References

Ayob, A. & Yassin, R. M. 2017. A confirmatory factor analysis of the attitude towards mathematics scale using multiply imputed datasets. *International Journal of Advanced and Applied Sciences,* 4**,** 7-12.

Byrne, B. M. 2012. *Structural equation modeling with Mplus: Basic concepts, applications, and programming,* New York, Routledge, Taylor & Francis Group.

Chen, L., Bae, S. R., Battista, C., Qin, S., Chen, T., Evans, T. M. & Menon, V. 2018. Positive attitude toward math supports early academic success: Behavioral evidence and neurocognitive mechanisms. *Psychological Science,* 29**,** 390-402.

Di Martino, P. & Zan, R. 2009. ‘Me and maths’: towards a definition of attitude grounded on students’ narratives. *Journal of Mathematics Teacher Education,* 13**,** 27-48.

Hannula, M. S. 2012. Exploring new dimensions of mathematics-related affect: Embodied and social theories. *Research in Mathematics Education,* 14**,** 137-161.

Hu, L. T. & Bentler, P. M. 1999. Cutoff criteria for fit indexes in covariance structure analysis: Conventional criteria versus new alternatives. *Structural Equation Modeling: A Multidisciplinary Journal,* 6**,** 1-55.

Kiwanuka, H. N., Van Damme, J., Van Den Noortgate, W. & Reynolds, C. 2020. Temporal relationship between attitude toward mathematics and mathematics achievement. *International Journal of Mathematical Education in Science and Technology***,** 1-25.

Kline, R. B. 2016. *Principles and practice of structural equation modeling,* New York, The Guilford Press.

Lipnevich, A. A., Preckel, F. & Krumm, S. 2016. Mathematics attitudes and their unique contribution to achievement: Going over and above cognitive ability and personality. *Learning and Individual Differences,* 47**,** 70-79.

Mcleod, D. B. 1992. "Research on affect in mathematics education: a reconceptualization". *In:* GROUWS, D. A. (ed.) *Handbook of research on mathematics teaching and learning: a project of the National Council of Teachers of Mathematics.* New York: MacMillan Publishing Co, Inc.

Muthén, B. O. 1989. Latent variable modeling in heterogeneous populations. *Psychometrika,* 54**,** 557–585.

Primi, C., Bacherini, A., Beccari, C. & Donati, M. A. 2020. Assessing math attitude through the attitude toward mathematics inventory – Short form in introductory statistics course students. *Studies in Educational Evaluation,* 64.

Suh, Y. 2015. The performance of maximum likelihood and weighted least square mean and variance adjusted estimators in testing differential item functioning with nonnormal trait distributions. *Structural Equation Modeling: A Multidisciplinary Journal,* 22**,** 568-580.

Yavuz Mumcu, H. & Cansız Aktaş, M. 2020. Development of an attitude-towards-using-mathematics scale for high-school students and an analysis of student attitudes. *International Journal of Mathematical Education in Science and Technology,* 51**,** 3-25.

Zakariya, Y. F. 2020. Investigating some construct validity threats to TALIS 2018 teacher job satisfaction scale: Implications for social science researchers and practitioners. *Social Sciences,* 9.

Zakariya, Y. F. 2021. Self-efficacy between previous and current mathematics performance of undergraduate students: an instrumental variable approach to exposing a causal relationship. *Frontiers in Psychology,* 11**,** 1-11.

Zakariya, Y. F., Nilsen, H. K., Bjørkestøl, K. & Goodchild, S. 2020. "Impact of attitude on approaches to learning mathematics: a structural equation modelling approach". *In:* Hausberger, T., Bosch, M. & Chelloughi, F. (eds.) *Third Conference of the International Network for Didactic Research in University Mathematics.* Bizerte, Tunisia: University of Carthage and INDRUM.

Zan, R., Brown, L., Evans, J. & Hannula, M. S. 2006. Affect in Mathematics Education: An Introduction. *Educational Studies in Mathematics,* 63**,** 113-121.

Gamification in the study of mathematics for engineering students

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Abstract

Gamification is an emerging approach to teaching that facilitates learning and promotes motivation through the use of game elements, mechanics, and game-based thinking. We used escape room as a gamification element in our mathematics course for engineering students to motivate them and improve the efficiency of formative assessment in their collaborative learning process. The Escape Room was implemented as a blended learning activity in the LMS, where students could only proceed with the quiz to the next question if they had answered the previous question correctly. Our approach is based on socio-constructivist learning theory and integrates collaborative and blended learning, formative assessment, and gamification. Our initial evaluation showed that the students accepted the gamified formative assessment well and that the final results in this course were better than before the use of the Escape Room.

Introduction

The development of digital technology as well as developments in the field of learning theories provoke changes in education. Deep, student-centred learning, where students are active and acquire knowledge at higher taxonomic levels, can be effectively supported by digital technology that enables students to explore and create their own learning materials, visualise and animate them, search, store and organise relevant multimedia resources, collaborate with peers and teachers, and organise learning activities in virtual learning environments. The concept of blended learning allows students to use both types of learning, traditional on-site classroom learning and e-learning with computer-mediated activities related to content and learning activities in online digital virtual classrooms implemented using a learning management system.

We present an example of a theoretically grounded and practically validated blended learning activity in a university mathematics course. We developed the blended learning activity to encourage our students to learn regularly and to improve the efficiency of their learning.

As we wanted to motivate them additionally to collaborative learning and ongoing formative assessment, we enhanced our learning activity with gamification elements. We placed the ongoing formative assessment in the *escape room*. *Escape rooms* are live action games where a group of players is trapped in a room and must collect clues and solve puzzles to escape. In our Moodle learning environment, *Escape Room* has been implemented as a quiz integrated into a classroom activity where students must solve specific tasks and select the correct answer to escape (Taraldsen et al., 2020). It is used as an on-site collaborative activity where students are organised in small groups.

A motivated student is more willing to face a task, more focused on completing it, and more persistent in overcoming difficulties. A motivated student is also willing to sacrifice more time and effort to achieve learning goals (Reyes, Enfedaque & Gálvez, 2017).

In designing and implementing the learning activity described above, we have taken into account scholarly insights on socio-constructivist learning theory, collaborative learning, e-learning and blended learning, game-based learning and gamification, and formative assessment. In the following sections we will introduce these important foundations and principles of the developed learning approach and explain how they are actualized in our learning environment.

Socio-constructivist learning theory

Constructivism is a learning theory that explains that learners develop understanding and form meaning from their own existing knowledge base, actions, and individual experiences. Newly acquired information builds on previously acquired knowledge to "construct" a broader understanding. Consequently, knowledge conception is a result of active cognition. Learning is flexible and more persistent in an appropriate environment.

The individual learns through discovery as personal development precedes learning. The constructivist approach to learning is learner-centred as the theoretical focus of learning is on the student rather than the teacher.

According to social constructivist learning theory, learning occurs when an individual interacts with other individuals as they practise, verify, solidify, and improve their mental models through discussion and information sharing. In education, the cooperative model emphasises and encourages collaboration among peers to support and reinforce the learning experience. Increased levels of interaction ultimately lead to increased creativity, critical thinking, and knowledge building (Schell & Janicki, 2013). In addition, collaboration helps improve communication and listening skills. Simply put, learners need to interact with each other to achieve meaningful knowledge acquisition (Chametzky, 2014).

We can summarise the main features of constructivist pedagogy in the following points:

- Learning content should be relevant to the learner,
- Learning should take place in authentic and real-world environments,
- Content and skills should be construed within the framework of the learner's prior knowledge,
- Learning should involve social negotiation and mediation,
- Teachers have the role of guides and facilitators of learning,
- Teachers should encourage multiple perspectives and representations of content;
- Students should be assessed formatively to inform future learning experiences;
- Students should be encouraged to become self-regulatory learners.

Online virtual learning environments and materials should be prepared with all these requirements in mind. The role of the teacher is to prepare the necessary conditions for the learning process in the form of safe, positive, friendly and motivating learning environments, and to provide feedback to students that promotes a meaningful learning experience.

Collaborative learning

Collaborative learning is a process in which students interact in dyads or small groups to discuss concepts or find solutions to problems, respecting the potential and contributions of individual members. Educational researchers have found that through peer instruction, students teach each other by addressing misconceptions and clarifying misunderstandings. Research shows that active, social, contextual, engaged, and student-led learning leads to deeper learning. The benefits of collaborative learning include developing higher-order thinking, oral communication, self-management, and leadership skills; increasing student retention, self-esteem, and sense of responsibility; learning about and understanding different perspectives; providing more opportunities for personal feedback; preparing students for real-world social and professional situations; and promoting student-teacher interaction.

Authority and responsibility are shared for group actions and outcomes, and interdependence among group members is encouraged. Collaborative learning changes the dynamics of the classroom by requiring discussion among learners. In learning environments that support collaborative learning, the learner is an active participant in learning rather than a passive recipient of education from an expert source (Udvari-Solner, 2012).

We based our learning activity on informal learning groups, which are temporary groupings of students formed spontaneously in the context of a class session. Students check their understanding, compare ideas, solve problems, and respond to questions.

Blended learning

When implementing an environment for online learning, it is very important to provide the infrastructure (e.g. LMS system), the learning content and the appropriate support for learning. For creating e-learning and blended learning course activities, experts in the field of educational sciences suggest implementing constructivist and socio-constructivist learning theories (Horton, 2000; Dabbagh & Kitsantas, 2005). It is very important to be aware of the learner's abilities as well as his previously acquired knowledge and stimulate his activity.
According to Vygotsky, the process of learning and cognitive development depends on social interaction. Therefore, students should cooperate with the teacher and with each other. In addition, learning can be enhanced through "scaffolding" where the teacher provides support to facilitate the learner's development. Appropriate blended learning is learner centred.
The original use of the term 'blended learning' was often associated with linking traditional classroom activities with e-learning activities. However, the term has evolved to encompass a much broader set of learning dimensions, such as blending online and on-site learning, blending independent and collaborative learning, or blending learning and practice (Bele & Rugelj, 2007).
In our approach to learning, we use the term to describe learning that combines controlled online delivery of tasks for formative assessment, on-site collaborative group activity by students to solve mathematical problems, and online recording of results.
This learning activity may be asynchronous for groups, but group members collaborate synchronously in problem solving using real-time face-to-face conversations. They share the same learning objective.
Learning motivation is increased when groups solve problems simultaneously because a competitive spirit is created. There is usually no interaction with the tutor. The learning process is governed using flow control and conditional commands available in the LMS activities. In this way, we were able to implement an *escape room* where students' learning activities were controlled. They were forced to solve tasks one by one, and they had to solve all tasks because otherwise they could not escape further.

Formative assessment

Effective formative online assessment can promote a learner- and assessment-centred focus through formative feedback and increased learner engagement with valuable learning experiences (Gikandi, Morrow, and Davis, 2011). Ongoing online formative assessment can improve the efficiency of student learning and reduce student anxiety (Lowe, 2015). Students can learn by making mistakes without feeling bad about having a wrong answer, as can happen with other types of formative assessments. No one else can see their mistakes, which can become a good learning opportunity this way. Quizzes are non-threatening and all students receive a grade. When students take quizzes frequently, they learn more, self-efficacy increases, and test anxiety is reduced. (Škapin-Rugelj& Rugelj, 2018).
Formal or informal class discussions about quizzes often uncover student misconceptions. Quiz test questions need to be academically sound, authentic, and important, similar in format and style to questions in exams (Snooks, 2004). The quiz activity module in Moodle allows the teacher to design and create quizzes consisting of a variety of question types. The questions are stored in the question bank and can be reused in different quizzes. The teacher can quickly analyse in which topics the students are successful and in which areas they have learning gaps. Teachers can use the graphical analysis of the online quiz to see if there are learning gaps in

the class. Such real-time data improves the formative assessment process. (Škapin-Rugelj& Rugelj, 2018).
We integrated quizzes into the gamified learning environment in Moodle LMS as key elements of Escape Room.

Gamification and game-based learning

Game-based learning can be defined as "learning that is facilitated by the use of a game." This can be at any academic level from preschool to lifelong learning and helps achieve learning goals at different taxonomic levels, from simple memorization and recall to high-level goals such as assessment or creativity. The use of play can be intrinsic or supplemental, face-to-face with physical objects or played online, with a computer (Whitton, 2012). Since implementing game-based learning is relatively challenging due to the time required for implementation and the limited availability of suitable serious games, teachers more often use gamification to make learning more engaging and efficient. It is a less demanding approach but still yields relatively good results.
Kapp (2012) defined gamification as an emerging approach to teaching that facilitates learning and promotes motivation through the use of game elements, mechanics, and game-based thinking. The student in a gamified learning environment does not play a complete game, but participates in activities that incorporate elements from games, such as earning points, overcoming a challenge, or earning badges for completing tasks (Varannai, Sasvari & Urbanovics, 2017). Our main idea was to integrate game-based elements into learning environments. Although this approach is essentially technology-independent, technology actually provides the most effective ways to implement it.
In addition to game elements, there are also a number of game dynamics or actions that take place while a player is involved in a game. These include merging, gathering, allocating resources, strategizing, building, solving puzzles, exploring, and role-playing. Combining these dynamics with the above game elements provides a context in which to engage learners through gamification. The idea is to gather the most effective game elements from a learning perspective and use them to motivate and engage learners (Chung, Shen & Qui, 2019).
Our students usually study for exams using their notes. When they encounter a problem, they usually consult with their peers and rarely seek help from teachers. To make exam preparation more efficient and fun, we introduced some gamification elements into the online learning environment. We designed and implemented an Escape Room as a quiz in the Moodle environment. The increasing popularity of escape rooms as entertainment and the popularity of learning games led us to implement the above learning activity as an Escape Room. Research on the impact of Escape Rooms as a learning activity has shown a positive impact on content knowledge and collaboration skills (Bartlett & Anderson, 2019).
The questions in the quiz were varied in nature and were designed to test students' understanding of concepts and computational skills. Students divided into groups to compete against each other. If a team manages to solve a certain number of problems within a certain time, they are rewarded with bonus points. With these points, students can improve their grades in the exam. Students first try to solve the tasks on their own, but if they get different results or do not find a solution, they discuss the problem in the group until they find a solution.

Method of investigation

The formative assessment with *escape room* was tested in the Mathematics 3 course, which covers ordinary and partial differential equations. The course is part of the undergraduate programme Water Science and Environmental Engineering at the Faculty of Civil Engineering and Geodetic Engineering at the University of Ljubljana. Instead of preparing for the midterm exam by reviewing the material by solving the problems on the blackboard, we decided to by

using the *escape room*. We set up the *escape room* as a quiz in Moodle, where the questions had to be answered in a specific order. Students have to answer the question correctly before they get the next question. The reason for this was that they solve problems as a group and discuss with each other about the topic.
The material covered in the iteration was linear differential equations and systems and Fourier series. The students were divided into two groups, one with 3 students and the other with 4 students. The quiz contained 10 questions of different types. Most of the questions were multiple choice questions and numerical questions where the correct result had to be write down. However, the quiz also contained some theory questions. The students all had notes with them. The groups approached the task differently. For some questions, everyone tried to solve the task on their own, and only at the end were the results compared. If they did not reach the same results, they discussed beforehand what could be wrong. On the other hand, questions that were more theoretical in nature were discussed beforehand. However, if they did not arrive at the correct solution together, the assistant helped them with an additional explanation. The students knew that if they solved the quiz in time, they would get a prize. However, they did not know what prize was waiting for them and that both groups would get a prize if they solved the quiz on time. This further fuelled the competitive spirit between the groups. Both groups solved the quiz ahead of time and received 5 bonus points, which were added to the points they had scored in the midterm exam.

Findings and discussion

We collected data for our empirical study through a survey distributed to 7 second-year student in the *Mathematics 3* course. An anonymous online survey based on Likert-type and open-ended questions was used to collect quantitative data and feedback on student satisfaction and the usefulness of the *escape room* in preparing for a midterm exam. 6 out of 7 students validly participated in the survey.
We asked students about the preferred number of team members. A few were of the opinion that it would be best to work in pairs, and most thought that the best groups would be with 3 members each. When we observed students solving a problem, we got the impression that in a teams with 4 members not all of them were equally involved. The quantitative results of the survey are presented in a graph using a Likert scale (1-strongly disagree, 5-strongly agree).

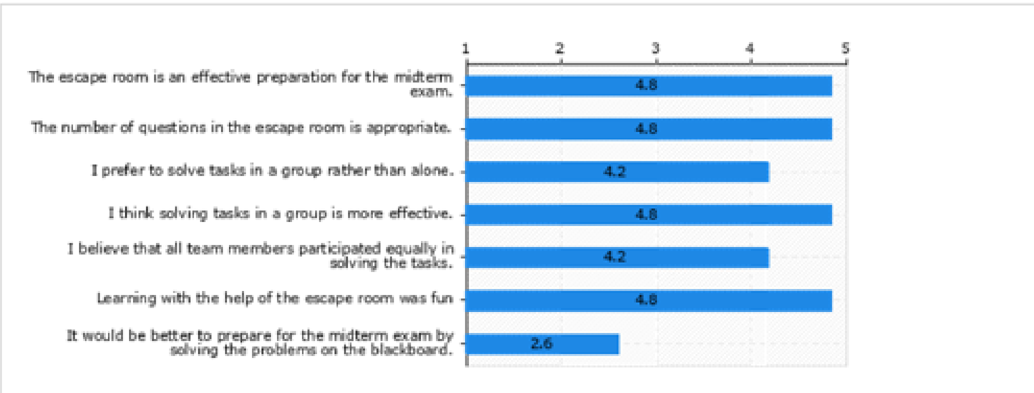

Figure 1. Average results of the survey

Here are some student opinions collected in the interview about the escape room as a way to prepare for the exam and their suggestions for improvements:

> "When preparing with the *escape room*, everyone is "forced" to think, brush up on the material, and solve a problem, which I think contributes more to learning than copying the assignment off the blackboard."

"I think that in the *escape room*, each individual is more forced to brush up on the material, to find a similarly solved problem, than if a single person solves the tasks on the blackboard. I think that in this type of solving, each individual learns much more." "I suggest that a group should have only two members, because in my opinion that is how everyone really works." "Everything was great."
"The *escape room* was a good test if one is prepared well for the midterm exam. Also, the extra percentages on the midterm exam and reinforcing what you didn't do so well on the quiz itself made you more motivated."

Conclusions

The results of our evaluation showed that students responded well to the use of gamification for formative assessment in a blended learning environment and that they were more motivated to learn and that they achieved better results from on-site collaborative learning. Therefore, we will continue to develop and apply gamified learning approaches in other mathematics courses.

References

Bartlett, K. & Anderson, J. (2019). Using an Escape Room to Support the Learning of Science Content. In K. Graziano (Ed.), *Proc. of Society for Information Technology & Teacher Education International Conference*, Las Vegas: Assoc. for the Advancement of Computing in Education. pp. 710-715.

Bele Lapuh, J., Rugelj, J. (2007) Blended learning-an opportunity to take the best of both worlds, *International Journal of Emerging Technologies in Learning (iJET)* 2 (3).

Chametzky, B. (2014). Andragogy and Engagement in Online Learning: Tenets and Solutions. Creative Education, 5. pp. 813-821.

Chung, C.H., Shen, C., Qiu, Y.Z. (2019) Students' acceptance of gamification in higher education, *Int. J. Game-Based Learn* 9(2*)*.

Dabbagh, N., Kitsantas, A. (2005) Using web-based pedagogical tools as scaffolds for self-regulated learning, Instructional Science 33. pp. 513-540

Gikandi, J. W., Morrow, D. and Davis, N. E. (2011) "Online Formative Assessment in Higher Education: A Review of the Literature, *Computers & Education* 57 (2). pp.2333-2351

Horton, W. (2000) Designing Web-Based Training, New Jersey:John Wiley & Sons.

Kapp, K.M. 2012. The Gamification of Learning and Instruction: Game-based Methods and Strategies for Training and Education, (1st. ed.). San Francisco:Pfeiffer & Company.

Lowe, W.T. (2015) "Online quizzes for distance learning of mathematics", *Teaching Mathematics and its Applications: An International Journal of the IMA*, 34 (3). pp. 138–148.

Reyes, E., Enfedaque, A. and Gálvez, J. C. (2017) "Initiatives to Foster Engineering Student Motivation: A Case Study",Journal of Technology and Science Education, 7 (3). pp. 291-312 .

Schell, G.P., & Janicki, T.K. (2013). Online Course Pedagogy and the Constructivist Learning Model. *Journal of the Southern Association for Information Systems*, (1)1.

Snooks, M.K. (2004) Using Practice Tests on a Regular Basis to Improve Student Learning. In Achacoso, M.V. and Svinicki, M.D., Eds. Alternative Strategies for Evaluating Student Learning. *New Directions for Teaching and Learning*. No. 100. San Francisco: Jossey-Bass Publishers, pp. 109-113.

Škapin-Rugelj, M., Rugelj, J. (2018) Using quizzes on a regular basis to motivate and encourage student learning. V: Proceedings SEFI MWG 2018. Coimbra: Coimbra Polytechnic - ISEC, The Department of Physics and Mathematics. pp. 45-50,

Taraldsen, L.H., Haara F.O., Skjerdal Lysne, M., Reitan Jensen, P. & Jenssen, E.S. (2020) A review on use of escape rooms in education – touching the void, Education Inquiry.

Udvari-Solner A. (2012) Collaborative Learning. In: Seel N.M. (eds) Encyclopedia of the Sciences of Learning. Boston:Springer.

Varannai, I., Sasvari, P., Urbanovics, A. (2017) The Use of Gamification in Higher Education: An Empirical Study, *Int. J. Adv. Comput. Sci. Appl.* 8(10).

Whitton, N. (2012) Collaborative Learning. In: Seel N.M. (eds) Encyclopedia of the Sciences of Learning. Boston:Springer.

The Role of Visualization in Mathematics

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Abstract

This paper presents some of new challenges in teaching mathematics by introducing active learning methods, unconventional learning scenarios and innovative instructional materials utilising dynamic visualizations and animations. Dynamics opens way to discover connections, and to understand mutual dependencies, which might often be more important than a detailed fragmented knowledge. Development of dynamic models is also an inspiration how to utilize information technologies as a meaningful didactic tool, which not only attracts learners but also enables them to realize their own creative ideas. Active participation of students in educational process in interesting form of developing own visualizations can support better understanding and more positive approach to learning itself, which becomes more a discovery of dependencies and investigation of processes than memorizing of isolated facts and not connected data.

Introduction

Mathematics is one of those subjects, in which teaching requires illustrations of abstract theoretical concepts and relations in a suitable way on examples, with aim to help students in better understanding of connections and enable them to attain better insight to presented problems. Dynamic mathematical programs are educational tools enabling visualization of mathematical entities in the form of dynamic models. Direct interactive manipulations with models offer possibilities of heuristic approach in acquiring knowledge, while learning of theoretical data is directly connected with their practical application in the respective dynamic model. Development of model itself and also work with it during study require a new attitude to the role of teacher and student in the educational process, change of the form of educational environment and contents of the educational process.

Active learning methods and how to introduce these in engineering mathematics is a real pedagogical challenge. Mathematical subjects are mostly accepted as necessary assumption to be fulfilled in order to graduate successfully from some engineering study program, not as a useful tool and symbolic language or effective method for solving applied problems. Courses of basic mathematics are important for engineering studies in particular, because future engineers will be dealing with various situations requiring logical thinking, analytical reasoning and ability to transfer knowledge and generalise findings, in addition to quick calculation and decision-making skills, and experience with mathematical modelling of applied problems. Therefore, it is necessary to prepare engineering students for their future work already during their university studies.

Visualization can be regarded as a certain form of application; therefore development of visual models is also a kind of evaluation of the knowledge depth and level of understanding of the presented concept, fostering acquired knowledge and its usage, and its transfer into a different context. Development of dynamic models is also an inspiration how to utilize information technologies meaningfully in the role of didactic tool, which can not only attract learners, but also enable them to realize their own creative ideas. Both subjects of the educational process act in this didactic situation more as equal partners, opposite to the classical forms of didactic situations, where the role of teachers is active presentation of new facts and data, while role of learners is usually passive, just receiving presented facts. Active participation of students in educational process in interesting form can contribute to better understanding and more positive approach to learning itself, which becomes more a discovery of dependencies and investigation of activities and processes than memorizing of scattered facts and not connected data. Dynamics opens way to discover connections, and to understand mutual dependencies, which is often more important than a detailed fragmented knowledge.

Visualization

The importance of visualisation and graphical representations in mathematics was demonstrated by several studies at the end of the last century, e.g. Bishop (1973,1980), Krutetskij (1976), Lean, Clements (1981) or Presmeg (1997, 2001, 2006). Battista (2007) states that spatial reasoning, as the ability to "see", explore and think about objects, ideas and relationships, is fundamental to geometric thinking. Based on this assertion, many mathematicians and maths teachers, and also psychologists consider visualisation to be an important part of mathematical thinking. As stated in Arcavi (2003), “visualisation offers a method of seeing the unseen”. Regarding the educational field, the role of visualisation has considerably changed recently, due to the increased development of technology. Various new possibilities have opened up for the teaching process with the emergence of dynamic mathematical programmes, updated facilities available in computer algebraic systems and provided graphical tools in geometric environments. Many researchers were motivated to study impact of visualization and then designated it by different names, such as mental imagery, visual mental imagery, visual imagery, visualisation or spatial visualisation. Some authors do not distinguish between visualisation and visual thinking. Mental imagery and visual perception are not identical, but they are functionally equivalent. It means that while visual perception occurs if “a stimulus is being viewed, and includes functions such as visual recognition and identification,” Gutiérrez (1996, 2006), visual mental imagery is a “set of representations that gives rise to the experience of viewing a stimulus in the absence of appropriate sensory input”, Kosslyn (2005). Zimmermann, Cunningham (1991) describe visualisation as the “process of forming images (mentally, or with pencil and paper, or with the aid of technology) and using such images effectively for mathematical discovery and understanding”. Zaskis, Dubinsky, Dautermann (1996) represent visualisation as a "means of transport" through which we can "travel" between the external context (external representations) and the mind, where an interaction between mental and external representations exists. Arcavi’s (2003) definition of visualisation states: “Visualisation is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings.”

Research in the domain of cognitive psychology, Moreno at all (2008), shows that human brain stores knowledge in two forms: in the graphical form as images, or in the form of words. It has been documented that methods of education, which are oriented to development of both of these forms of knowledge acquisition, in a considerable way influence quality and depth of understanding and sustainability of acquired knowledge. Direct inclusion of learners into the development and utilisation of non-linguistic representations within the process of learning considerably stimulates and increases brain activity. This further leads to the development of cognitive connections, which consequently foster knowledge and deepen understanding of basic principles and concepts. Manipulative techniques and tools are concrete or symbolic artefacts that students directly use while obtaining new pieces of knowledge. These powerful didactical tools enable active, hands-on explorative methods and heuristic investigation of abstract rules and relations. Research results prove that computer aided manipulative techniques and tools are even more effective than usage of physical models, because they can dynamically connect several possible representations and interpretations of studied constructs and relations. Dynamic working sheets developed in the dynamic mathematical programs provide users with possibilities to define dynamical mathematical objects, as e.g. graphs of functions, polygons, curve segments, etc. and interactively manipulate with them. These mathematical objects are real objects of the virtual platform, and we can therefore speak about continuous dynamic interaction between these objects and users. Dynamic mathematical software can be regarded as manual equipment detecting motion of slider and consequent changes of the depicted objects, as presented also by Karadag, McDougall (2009), Velichová (2011, 2018).

Active Learning Methods

The concept of active learning appeared in the early 90-ties. Main point of this concept was to put more focus on the learners and their needs, although the term covers today a broad spectrum of different approaches, as usage of new technology, collaborative learning, project-based learning, learning-by-doing, and others. The main idea of active learning implementation and its main drive is to put the responsibility of learning at the hands of the learners themselves. As the exact opposite of passive learning, which distributes the roles between instructors and lectures who deliver information and students who merely memorize and assimilate, active learning changes the role of teachers to act as guides who let the learners learn on their own through the use of different active training activities. Benefits of active learning have been studied and analysed in many experiments, and published in a plethora of studies. To mention just a few of them: increase of content knowledge, development of critical thinking and problem-solving abilities, creative thinking, imagination, collaborative and interpersonal skills, while according to Andriotis (2018), learner motivation remains one of the most important from all. It has been proved by the everyday evidence that implementation of active learning methods in teaching increases enthusiasm of both learners and facilitators. And even more, active learning also improves learners' perception and attitude towards the entire learning process, strengthens their motivation and raises their interest and involvement into and desire to acquire new knowledge. These are all critical attitudes in establishing an active learning environment, Pinto at all (2018).

Among a number of well-established active learning activities we will introduce one of them, eduScrum, in details. Scrum methodology is a well-known framework for project management, and eduScrum is its adaptation to education. EduScrum, like Scrum, is founded on empirical process control theory – empiricism. Empiricism asserts that knowledge comes from experience and making decisions based on what is known. EduScrum employs an iterative, incremental approach to optimize the achievability of learning goals and control risk. Implementation of this method needs a lot of preparatory work, which means a thorough didactical plan and strategy for the whole period of implementing this method for teaching in particular courses, and development of didactic materials to be used during collaborative activities in the classroom, Sutherland (2014, 2015). The main issue is to prepare a detailed timetable of special activities distributed throughout the course run, during which students cooperate in small groups, usually 4 – 5 persons, and solve as a team prepared collection of problems that are related to particular educational material unit, called sprint. Number of problems to solve might vary, but usually there is one problem for each team member. Students´ team works independently; members discuss all tasks together, while one of them always acts as a group leader – scrum master, while this role circulates among all students in the group during the course. Group leader distributes task problems to members of his team, and s/he is responsible to check and correct their solutions, to collect them all and finally to report them to the rest of the students in the class. All problems can be thoroughly discussed with other groups in the class and the whole activity can be evaluated. Small teams work in addition on projects related to their tasks solved within particular sprints. These projects are tasks to visualize dynamically some selected problems and present these developed dynamic visualizations to the whole group of students in the class. Students from other small groups evaluate presented visualizations while each member of the presenting group is awarded the same point score for the particular sprint visualization.

Conclusions

There can be identified many positive aspects of introduction of dynamic visualizations, active learning methods, and eduScrum method in particular. To mention some of these, continuous assessment of students´ knowledge during whole semester and from separate parts of curricula should be mentioned at first. In addition, students learn how to work in a team, while they have to be responsible for their particular tasks, and for the whole team as a unit. They are trained to lead constructive discussions about problems, while through these discussions they might get better insight into various solution strategies, to acquire new knowledge and understand better the core of the problems while finding their visualizations and their applications. Generally, students claimed in various opinion polls that through these active learning scenarios they could learn how and "why" they should improve their knowledge from mathematics.

There can be pointed also several negative aspects of eduScrum method. Random distribution of students to working teams often based on their personal relations and friendship after the arrival to university study, not on the base of the mathematics knowledge level might cause problems. Weaker students appreciate help from more skilled students with solutions of their attached problems, and thanks to better students they acquire more points in their "score" from particular topics. This does not motivate them to learn how to "grasp" a problem, how to start to solve it, nor how to cooperate more actively and independently. Sometimes, better students could be "constrained" by weaker students. They have to explain to weaker students how to solve assigned problems, and in order to receive as much points as possible for their own score they solve entirely also problems of other students, with the same points awarded to everybody in the team. There were also situations that they received less points on their "score" for not correct solutions of problems solved by weak students, as they could receive provided they had solved the problems themselves and correctly, or in another constitution of the working team. Better students were sometimes willing to work in smaller groups (2 – 3 members) and they agreed on finding solutions of all 5 assigned problems within the given time limit, as analysed in Velichová, Gabková (2019).

Finally, it is necessary to admit that a pure ability to plot and graphically interpret mathematical object does not mean itself any full understanding of the mathematical relations geometrically represented by the generated visual image. Dynamic work with these objects and heuristic investigations are necessary to acquire more comprehensive understanding and to develop abilities to use technologies for analysis and solution of problems. This is fully consistent with the general understanding of mathematical knowledge as a sustainably increasing ability to utilise various representations of mathematical concepts in different contexts, exchange them dynamically according to actual needs, and be able to illustrate and apply these concepts correctly.

References

Andriotis, N. (2018) "Active Learning Methods for Super Engaged Corporate Learners". https://www.efrontlearning.com/blog/2017/05/active-learning-methods-engagedcorporate-learners.html.

Arcavi, A. (2003). "The role of visual representations in the learning of mathematics". In Educational Studies in Mathematics, vol.52, no.3, p.215-241.

Battista, M. T. (2007) "The development of geometric and spatial thinking". In F. Lester, (Eds.), Second Handbook of Research on Mathematics Teaching and Learning. United States in America: Information Age Publishing Inc, 2007, p. 843-908.

Bishop, A. J. (1973) "Use of structural apparatus and spatial ability: A possible relationship". In Research in Education. 1973, vol. 9, issue 1, p. 43-49.

Bishop, A. J. (1980) "Spatial Abilities and Mathematics Education: A Review". In Educational Studies in Mathematics, vol.11, issue 3, p.257-269.

Gutiérrez, A. (1996) "Visualization in 3-dimensional geometry: In search of a framework". In L. Puig & A. Gutierrez (Ed.), Proceedings of the 20th PME International Conference. Vol. 1. Panama City, Florida, p. 3-19.

Gutiérrez, A., Boero, P. (2006) "Handbook of Research on the Psychology of Mathematics Education. Past, Present and Future". UK: Sense Publishers, 2006. 61 p.

Karadag, Z., McDougall, D. (2009) "Dynamic worksheets: visual learning with the guidance of Polya", In: MSOR Connections, Vol. 9, No 2.

Kosslyn, S. M. (2005) "Mental images and the brain". In Cognitive Neuropsychology, vol. 22, no.3/4, p. 333 – 347.

Krutetskij, V. A. (1976) "The psychology of mathematical abilities in schoolchildren". Chicago, IL: University of Chicago Press, 1976. 417 p.

Lean, G. A., Clements, K. (1981) "Spatial ability, visual imagery, and mathematical performance". In Educational Studies in Mathematics, vol. 12, no. 3, p. 267-299.

Moreno, A., Armella, L., Hegedus, S. J., Kaput, J. J. (2008) “From static to dynamic mathematics: Historical and representational perspectives”. In Educational Studies in Mathematics, Vol. 68: 99-111.

Pinto C., Nicola S., Mendonca J., Velichová D., Heler A. (2018) “New Methodologies for Teaching Math Courses in Engineering Degrees”, The 19th SEFI MWG Seminar on Mathematics in Engineering Education, proceedings, Coimbra 2018, Portugal, pp.70-77.

Pressmeg, N. C. (1997) “Generalization using imagery in mathematics”. In L. D. English (Ed.), Mathematical Reasoning: analogies, metaphors and metonymies in mathematics learning. Mahwah, New Jersey: Lawrence Erlbaum Assoc., p.299 - 312.

Pressmeg, N. C., Balderas-Canas, P. E. (2001) “Visualization and affect in non-routine problem solving”. In: Mathematical Thinking and Learning, vol. 3, no. 4, p. 289 – 313.

Pressmeg, N. C. (2006) “Research on visualization in learning and teaching mathematics”. In A. Gutiérrez & P. Boero (Ed.), Handbook of research on the psychology of mathematics education: Past, present and future. Rotterdam, The Netherlands: Sense Publishers, p. 205-235.

Sutherland, J. V. (2014) “Scrum: the art of doing twice the work in half the time”. New York: Crown Business.

Sutherland, J. V. (2015) “The eduScrum Guide: The rules of the game“.

Velichová, D., Gabková, J. (2019) “EduScrum method in teaching mathematics to engineering students”. In: The 47th SEFI Annual Conference, Budapest 2019, Proceedings, SEFI Brussels, Belgium, pp. 1962-1971.

Velichová, D. (2018) “Dynamic Visualizations in Education of Mathematics,” In Proceedings of the SITE 2018 Conference, Washington D.C. USA, pp. 1858-1863.

Velichová D. (2011) „Interactive Maths with GeoGebra“ In JET - International Journal of Emerging Technologies in Learning, Vol 6 (2011): Special Issue: VU2009, pp.31-35.

Zazkis, R., Dubinsky, E., Dautermann, J. (1996) “Coordinating visual and analytic strategies: A study of students’ of the group D4”. In Journal for Research in Mathematics Education, vol. 27, no. 4, p. 435–457.

Zimmermann, W., Cunnigham, S. (1991) “Visualization in Teaching and Learning Mathematics”. Washington, DC, USA: Mathematical Association of America. 224 p.

Tools of Distance Learning - Expectations and Experiences

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Abstract. Recent transfer to distance education forced many teachers to reshape their courses and adapt new tools. In fact, many of these tools have been around for years, but they were not used for various reasons. The new experience led to some re-appraisal of their perceived advantages and disadvantages and the way that they can be applied.

1. Introduction

Most countries around the globe experienced lockdowns recently, and university education was a prime target. In fact, many universities moved to distance education voluntarily, without a government directive. Educators thus had to cope with a new situation that only a few were prepared for.

However, the concept, practice and tools of distance learning have a long history going back, in its modern forms, to the 1940s. The need to serve off-site students came from various motivations. It could be inspired by the need to serve people in scarcely inhabited countries with a large proportion of people living in rural areas, as it happened for instance in Australia, or it could be aimed at people in special circumstances, like the distance education programs serving military personnel overseas and programs serving prison inmates in the USA. Another possible motivation was to allow access to higher education to people whose socioeconomic background made it difficult to attend a regular university or who wished do extend their education later in life, as was the case with the Open University in Britain founded in 1969 [OU].

As a result, quite a lot has been written about distance learning and various tools. Some results were more prominent, for instance the The Seven Principles of Good Practice [Bangert], many results can be found in journals on education (for instance [HannayNewvine]), and many more results are strewn across many specialized journals (for instance [NielsenSheppard], [CallasEtAl], or [BockEtAl]).

Yet, as it is often with research in education, relatively very few educators were actually aware of these contributions. The attitudes of majority of educators, who did not see the research and had no real practical experience with distance learning tools, were usually based on feelings and extrapolation of one's experiences. This de_nitely does not mean that the attitudes were wrong, as an experienced educator can often make a good assessment of new approaches and methods; on the other hand, it does happen sometimes that educational experience is not enough to foresee unexpected side-effects of various measuresand approaches.

For the past year, we have been essentially forced to adopt many tools, and we now have some personal experience with their effectiveness, and with their positives and negatives. Moreover, now we also have a signiffcant student feedback. Were there any surprising facets appearing, or facts that would make us re-appraise some tools? We will now look at the basic components of a typical high enrollment course at a technical university and record our observations.

2. Lectures

Lectures delivered off-site have a long history behind them. The first generations of distance learning programs were correspondence courses, and it was soon recognized that such courses suffer in comparison with traditional programs, with a significant difference observed between students learning just by reading supplied materials and students that were given presentation. As technology progressed, it was readily adopted to distance learning. Lectures were broadcast by radio, then broadcast by TV and eventually delivered as taped lectures to play on VCR. Already in those days researchers were assessing the new tools, and many of their findings are still of interest today.

At first, technology constrains meant that taped and distributed lectures were created only by teachers participating in special programs. But that changed around the turn of the century, culminating in the current situation when a reasonably tech-savvy teacher can prepare videolectures and place them on Internet without much problem. Moreover, the educators in general became aware of such possibilities. However, only few chose to do so. What were the reasons for and against? Possible reasons for taping lectures:

- Some institutions saw on-line courses as a marketing tool, and actively pursued this direction.
- Taped lectures are useful for students who for some reason miss a lecture.
- A taped lecture, stored safely in a university vault, is a nice thing to have and play to students in class when a lecturer falls suddenly ill.
- A taped lecture is a very good form of a feedback for a lecturer, as the student point of view of a lecture can reveal many aspects to be improved, for instance regarding the size and legibility of writing, delivery, organization of presented material and other aspects of one's teaching.

Perceived negatives:

- A taped lecture is not alive, and therefore it lacks important features like the feeling of personal connection between the lecturer and the attendees, engagement in the topic, and last but not least, community building.
- A taped lecture is not interactive.
- When a true live lecture is recorded, including student participation, the privacy of students is invaded. Also, students feel inhibited from asking questions when they know that the lecture is being recorded.
- Teachers often feel apprehensive about the possibility that their lectures, including small mistakes and their personal quirks, become widely available for students to dissect and pick out juicy bits. Experience with cyberbullying, in particular with students making fun of each other and their teachers through videos shot with cellphones, serves as a deterrent.
- If lecturers taped their lectures, they would no longer be needed.

Comparing negatives to positives from the point of view of a teacher, the small demand for lecture taping on the side of teachers is really not surprising. The validity and seriousness of some of these advantages and disadvantages have been addressed already in the previous years, and there are new observations being made in the lockdown era. Some of the findings include:

- Experience suggests that there is some drop in lecture attendance when a taped lecture is available, but the extent of that drop varies with the content of the live lecture. If the lecturer merely reproduces the textbook text, preferably in a monotone voice while looking into the distance, avoiding eye contact with students, then indeed, there is not much incentive attending a live lecture, and replacing such a lecturer with a tape or a robot may be a sensible move on the side of administration. However, a good lecturer offers more than just a reproduction of a written material. A richer content, personal commitment and enthusiasm about the subject, these are things that make quality lectures special and do not translate easily to recordings. Several surveys based on current lockdown student feedback report that a large percentage of students would prefer to return to live lectures.

- The student teaching evaluations at the end of the fall semester in 2020 at our institution included the question which aspect of the distance education would the students like to see preserved when the situation gets back to normal. The predominant answer (more that three quarters of responses) was to preserve availability of taped lectures. This is not in contradiction with the previous point. Students appreciate most if they can attend a live lecture and still have the option to look again at parts that were too fast or too involved to understand them at the _rst go. It also gives them a fall-back option in case they cannot make it to a lecture.
- The easiest way to produce lectures for remote students is a live streaming that is also recorded. The demand on technical services and time is much smaller compared to pre-taped and edited lectures, which makes this a natural way to run distance education. Another perceived advantage is that there might be at least some level of interaction. However, in a typical setting, achieving interaction is rather diffcult, so this advantage is often illusory. Moreover, quite a few teachers feel uncomfortable about such impromptu recordings, in particular when they feel that the delivery was not the best they can do, and ask administration to restrict access to them, even delete them at the end of the semester. From this point of view, pre-recorded lectures present quite a few advantages. A teacher may re-record parts that were not done to the best satisfaction, edit the recording (cutting out breaks for clearing the blackboard may shorten the recording by up to 10 percent), add subtitles, and improve the quality of the lecture in general. The teachers then tend to be more willing to publish their lectures. The drawback of this solution is that editing takes time and technical skills that not every teacher possesses.
- Numerous studies indicate that the effectiveness of recorded lectures does not differ signifcantly from live ones. Typically this is measured by comparing tests scores at the end of the course. This is definitely a good argument for taped lectures and shows that at least in some aspects, distance education may rival the traditional way in its results. However, there are some factors that have to be considered. First, such studies do not measure how much effort it took for students to actually reach their eventual level of knowledge. Second, merely giving students recorded lectures in place of live ones does not mean that comparable education will be taking place. Distance courses with good results typically feature a more thorough adjustment of course structure. In short, the time that a teacher saves by giving students recorded lectures (perhaps from the previous years) should be given back in other ways, namely by employing alternative educational strategies that will in some way create a personal bond and interaction between a student and the instructor.

An important note should be made. The observations above do not apply universally, as there is a wide range of interactivity required in different courses. A course for 20 students requiring close guidance and frequent interactivity would find it much harder to replace on-site teaching with distant forms, which also applies to lectures. On the other hand, a typical course on introductory mathematics (calculus, linear algebra, etc.) is typically attended by 200-400 students, and in such case the disadvantages of distance lectures become very small compared to advantages.

3. Seminars, practical classes

A typical course consists of material that can be reasonably well transmitted to a large number of students at once (traditional courses do it via lectures, alternative approaches like flipped learning do it through videos), and interactive parts where practical skills are acquired and students get some form of feedback. These could be, depending on the institution, called practical classes, reading classes, labs, and other names. A typical practical class hosts about 20 students, the general feeling being that this gives every student a reasonable share of the teacher's time and attention.

On the other hand, now and then an idea comes up that there could be some savings if such practical classes were joined to somewhat larger groups, for instance a course with 240 students could be split into groups of 80. This makes sense from administration's point of view, as one employee can serve a higher number of students. Obviously, for courses that are heavy on one-to-one interaction, the merging of classes is simply impossible, but in many courses, the practical classes feature students working on assigned problems, asking questions about them, and discussing outcomes. Then it really does not matter so much how many students one has in class.

In fact, the authors had a personal experience with such an arrangement, when an emergency situation forced us to teach so-called seminars instead of small practical classes. The eventual outcome did not seem bad, but we quickly returned back to small classes when it was possible, and the administration concurred. What were the arguments?

One argument is related to the fact that even when there is less interaction in a practical class oriented on problem solving, the teacher still wants to have the possibility to work with students individually if needed, which is just not possible in a class of 80, in fact practical experience shows that even 40 could be too much. The second argument is related to social aspects of teaching. Teaching 20 students in a small class feels distinctly different - more personal – compared to teaching 80 people in a large hall, which in turn has a visible impact on student's willingness to ask questions. Asking a question when sitting in the third row, within an eye contact with the lecturer, feels very much different compared to asking it while sitting in row 20, when the teacher is just a small fuzzy blob somewhere in the distance. In short, the small classes play an indispensable role in on-site education.

However, the distance education offered a somewhat different perspective. While looking for an organization of a course better fitted to distance teaching framework, both smaller and larger size practical classes were also tried. Here are some interesting observations:

- In a video conference, students rarely switch on their cameras. In a course taught by the authors in the fall 2020, only 13 students out of 180 would have cameras on routinely. This removes much of anxiety connected with asking questions. Still, many students did feel too intimidated to speak up, and for those the possibility to ask questions by writing them in a chat was a big help. While we do not have numbers on hand (unfortunately, we cannot go back in time and do statistics for on-site courses of the past), the number of questions asked during online practical sessions seemed higher compared to previous years, and the classes felt more alive. In short, on-line practical classes have the disadvantage of feeling less personal, but this actually allows students to feel more free to explore the topics, thus helping students in mastering the content.
- Teaching several instances of the same practical class shows that questions tend to repeat themselves to a large extent. Thus connecting four small classes into one seminar does not mean a fourfold increase in questions. However, the questions do not match exactly, so there is some increase. This is beneficial, because now all students in a large class get to hear the questions and the answers, further helping them in their understanding of the topic.
- It is not easy for a teacher to lead a practical class and at the same time keep an eye on the chat. Some questions remain unanswered for minutes, and often other students would step in and put some answers in the chat. These are usually correct, and students helping one another foster the team spirit.

In short, experience suggests that joining practical classes into larger groups has more advantages than disadvantages when organized as videoconferences with a chat. As usual, this is not a general rule, as the actual arrangement of particular classes varies and some versions are not as readily scalable as others. Moreover, when we do connect classes like this, we have to make sure that there will be another way to deliver personal feedback on how students do with their coursework. Which brings us to the third component.

4. Homeworks

Homeworks have been the part of education since the first proto-human sent a young one to bring a banana from the forest. As courses, institutions and educational systems differ, homeworks took on many different forms, from optional to obligatory, from ungraded or self-graded to those where solutions are presented and dissected in class.

The distance education served to put homeworks in sharper focus and increase their importance. The fact that feedback plays a key role in education has been long acknowledged, and teachers find different ways to implement it in their courses. However, it is not easy to find ways suitable to distance teaching, and the homework is one of the tools that do provide ready feedback on where students stand at the moment with their mastery of the subject. We are not aware of any new revelations, but the considerations that applied to homeworks before become even more important in distance education.

- Homeworks play an important role in guiding students' effort. When designing a course that is to be taught remotely, it is crucial to pay close attention to the design and contents of the homeworks. Merely throwing in some problems for practice is possible, but does not unlock the full potential of this educational tool. One may even use several homeworks for every week, each with different setup and rules, to shape students' approach to the course.
- A proper feedback is of utmost importance. Marking the work as correct or incorrect could do in a situation when a student has other possibilities for interaction with instructors. In a course taught remotely, a more thorough feedback is a must. Errors in calculations or reasoning should be pinpointed, and when a problem in understanding some key concept is identified, the student should be warned and either get a brief explanation when possible, or be pointed to some book or video.

Of course, such thorough commenting takes time and a lot of effort, but it could be offset by using pre-recorded lectures and joined practical classes. In smaller classes, individual online consultations over homework solutions could be used, and these work even better.

Again, here one's options can be severely restricted by outside considerations and also the course's content. Alternative strategies to increase students' afiliation with the course may be employed. One strategy that seemed to work well was to start every practical class by a (different) student presenting solution of a homework. In this way, all students became at least once an integral part of the teaching process. However, here a fine line has to be drawn so that students presenting their work should not feel that it is them who are being dissected, the focus must be on the material and on one of the basic rules of education: The more errors we make in practical classes, the less errors we will make at the exam.

Conclusion

The recent situation at universities forced teachers to reshape their courses and re-evaluate usefulness and use of some tools. Many people predict that the recent experiences will have a long-term impact even when (if) the situation gets back to normal. Given the newly perceived benefits of some approaches, this is very likely true.

First, we can expect that a number of teachers will not want to give up on tools that seemed to work in distance learning when on-site education returns. In particular, students now had to work more on their own, and it seemed to help them, but only when they obtained a good feedback and guidance. Can this somehow be incorporated in traditionally-structured education? This will be a big challenge for teachers who are true educators.

Second, and most likely more important, the students had their own experiences with various approaches, and they will deffinitely make their opinion be heard. In particular, we can expect a pressure to make lectures available online, and universities will have to find ways to accommodate interests of all participants in educational process. If this happens, then it will also force teachers to rethink how they shape their courses. Just coming to the lectures to repeat what is recorded seems like a waste, surely one can do better. How? That is another good question, and we will surely see some interesting ideas emerge, in particular one can predict an increased interest in flipped learning.

References

[Bangert] Arthur W. Bangert, The Seven Principles of Good Practice: A framework for evaluating on-line teaching. The Internet and Higher Education, Volume 7, Issue 3, 3rd Quarter 2004, Pages 217-232.

[NielsenSheppard] Eleanor Nielsen, Margaret A. Sheppard Television as a patient education tool: A review of its e_ectiveness. Patient Education and Counseling, Volume 11, Issue 1, February 1988, Pages 3-16.

[HannayNewvine] M Hannay, T Newvine, Perceptions of Distance Learning: A Comparison of Online and Traditional Learning. Journal of online learning and teaching, Volume 2, No. 1 (March 2006).

[BockEtAl] Thomas Brockfeld, Bringfried Mueller & Jan de La_olie, Video versus live lecture courses: a comparative evaluation of lecture types and results. Medical Education Online, Volume 23, 2018 - Issue 1.

[CallasEtAl] Peter W. Callas , Tania F. Bertsch , Michael P. Caputo , Brian S. Flynn, Stephen Doheny-Farina & Michael A. Ricci, Medical Student Evaluations of Lectures Attended in Person or From Rural Sites via Interactive Videoconferencing. Teaching and Learning in Medicine, Volume 16, 2004 - Issue 1.

[OU] Open University, United Kingdom, http://www.open.ac.uk/.

New Guidelines for the National Curriculum Regulations for Engineering Education in Norway

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Abstract

The Norwegian Bachelor of Engineering Education is based on National Curriculum Regulations given by the Norwegian Ministry of Education and Research. Revised Regulations were adopted on 18th of May 2018. Universities Norway (UHR) was given the task of developing and approving Guidelines for these Regulations. The curriculum Regulations and the Guidelines are based on the Norwegian qualification's framework for higher education. Guidelines are given on both knowledge, skills, and general competencies. UHR and Centre for Research, Innovation and Coordination of Mathematics Teaching (MatRIC) organized a workshop, with representatives from all institutions with engineering programs, to work on the update of the Guidelines for mathematics. The new Guidelines also include developments in digital competence, basic programming, and social sciences. This article presents National Curriculum Regulations in Norwegian Higher Education, and discusses Guidelines for national Curriculum Regulations, developing study programs according to Guidelines with emphasis on Mathematics, as well as some examples of implementation.

National Curriculum Regulations in Norwegian Higher Education

The Ministry of Education and Research (KD) issues directives on National Curriculum Regulations for certain higher education programs. Examples include teacher's education, health education and engineering education.

The rationale for national curriculum regulations is that for these professions, there is a need for national coordination of, among other things, academic content and learning outcomes. However, the regulations should not be more detailed than necessary, since they imply that the rules and requirements are mandatory, and they may represent a challenge for the autonomy of the institutions. Most educations that are regulated have traditionally been three-year and vocationally oriented, and they are offered at many higher educational institutions throughout the country.

Guidelines for National Curriculum Regulations

Guidelines for National Curriculum Regulations are recommended standards, prepared by the academic communities themselves, to assist the institutions in the planning of their own programs and curricula, and can be given different formal status. The guidelines for engineering education are anchored in the regulations, and the regulations must be met by the institutions. UHR, which is responsible for national coordination in the higher education sector was commissioned by KD to develop, agree on, and adopt national guidelines for engineering education in accordance with the regulations. The strategic body for the STEM-field in UHR (UHR-MNT) consists of deans or other leaders of all faculties or departments with STEM education and research. It is these leaders who have adopted the guidelines. That is, it is these frameworks, regulations with attached guidelines, the institutions must comply with. The guidelines are thus not voluntary but have room for flexibility and adaptation within the individual study programs. The guidelines are expected to be in continuous development through national cooperation.

There is a continuous discussion about the requirements for engineering education and their Guidelines set on a national level. Regulations and Guidelines are updated from time to time, in a collaboration between the Ministry of Education and Research, Universities Norway (UHR) and the individual institutions and academic staff as described in Jakobsen (2012) and Jakobsen et al (2017). The result is often a compromise between actors who want as much freedom and flexibility as possible in the design of their own engineering programs, and those who want a substantial common core of specific learning goals, which all study programs must include. Common to all actors is undoubtedly a goal to offer the best possible engineering education, but at the same time different perceptions of what is "best" and what an engineering education should be.

An issue is a reported experience of ambiguity about the function and status of the Guidelines. One possibility is that they are concrete mandatory requirements, defining the broader intentions of the Regulations themselves. This may be achieved by giving the Guidelines the same formal status as the Regulations and using the phrase "must include". Another possibility is that the Guidelines are perceived as recommendations and suggestions on how to fulfil the Regulations, that they are optional, and thus may be adapted or even ignored to suit the wishes of each institution, provided the Regulations themselves are met. This may be achieved by giving the Guidelines the formal status of recommendations and/or writing "should" and "can".

The Regulations and Guidelines together comprise the framework ("Rammeplan") defining the engineering education. In addition, the protected title of Sivilingeniør (Siv. Ing., not to be confused with the English concept of "Civil Engineer") is reserved for engineers that subsequently complete a 2-year master's degree in technology, and whose engineering bachelor's degree fulfils additional requirements on the volume of natural science and maths content. UHR-MNT was also given responsibility for preparing and adopting these conditions for using the title Siv. Ing. on the diplomas. The admission requirements for these masterprograms are an engineering bachelor's degree according to the regulations including 25 ECTS maths, 5 ECTS statistics and 7,5 ECTS physics. Many institutions choose to adopt these requirements, since the title Siv. Ing. is considered prestigious and lucrative in the job market. Figure 1 visualize the components and context for a Norwegian Bachelor of Engineering degree based on the national curriculum regulations and guidelines.

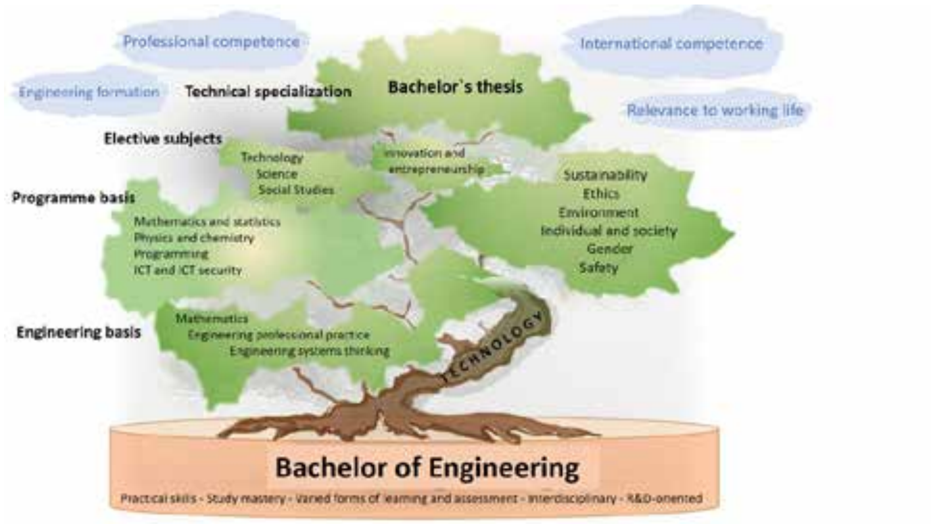


Figure 1. Components and context for a Norwegian Bachelor of Engineering degree based on the national curriculum regulations from May 2018.

We further describe the Guidelines pertaining to Mathematics below. Following the Guidelines ensures that the programme fulfils the national Regulations, but there is room for adjust both individual learning goals and volume of for instance mathematics (ECTS count) to suit the wishes of the individual institutions.

Developing study programs according to the Guidelines

Historically, what is known as the "natural sciences and mathematics" element of the Engineering programme has received considerable attention in the debate. This is because although it is clear that the engineering profession involves natural sciences and maths (defined here as maths, physics, chemistry, statistics and IT), some of these subjects may be perceived as irrelevant to specific engineering specialisations. One ambiguity that often appears in the conversation is the role of the natural sciences and maths:

- Are natural sciences and maths viewed only as service subjects whose function is to prepare students for the later learning goals in technical subjects and specialisation subjects? Does this mean that a specialisation, that does not in this way need a particular science subject, may omit it?
- Or should all engineering degrees include some minimal amount of all the natural sciences and maths, irrespective of whether they explicitly support later specialisation subjects? Is this an integral part of being an engineer?

In practice, this is of great importance for how the studies are designed, how the natural science and maths subjects are taught, to what extent and by whom. In three years, it can be a challenge to design a rounded study with a good transition from upper secondary level, which ensures the necessary academic breadth, reaches sufficiently deep into specializations and electives, and concludes with an individual project. Here, demands for content that is perceived as "unnecessary" can lead to frustration. Related issues are discussed by Kent and Noss (2002).

The program basis with emphasis on Mathematics

When it comes to Mathematics the Guidelines refer to the SEFI report “A Framework for Mathematics Curricula in Engineering Education”, Burkhard Alpers et. Al (2013), as a benchmark for content-related competencies, knowledge, and skills.

The preface of the mathematics Guidelines states the following:

> Mathematics and statistics are necessary tools in technological subjects. Complex and accurate mathematical calculations are an important part of technology development and innovation. Measurement, analysis, and decisions based on numerical data is involved. On the engineering education level, e.g defining all different engineering education programs, regardless of subject area, learning outcomes for mathematics, as knowledge, skills and general competence, are defined in the guidelines.

Then follows three Knowledge learning goals, seven Skills and three General Competencies, defined in accordance with the Norwegian qualification’s framework for higher education.

Specific mathematical topics are also defined:

Calculus, part 1: Functions, inverse functions, limits, continuity, derivation, integration, methods of integration, ODE’s and modelling.

Calculus, part 2: Complex numbers, functions of multiple variables, partial derivative, Laplace transform, series, power series, Fourier series, finite difference equations.

Linear algebra: Matrices, determinants, systems of linear equations, eigenvalues and eigenvectors.

Combinatorics, mathematical reasoning.

The mathematics Guidelines were written with the expectation that they would be mandatory. The intention was to collect current practice at the relevant institutions into a common set of learning goals and concrete mathematics topics. Since these were the de facto national standard, they could be formally defined as such.

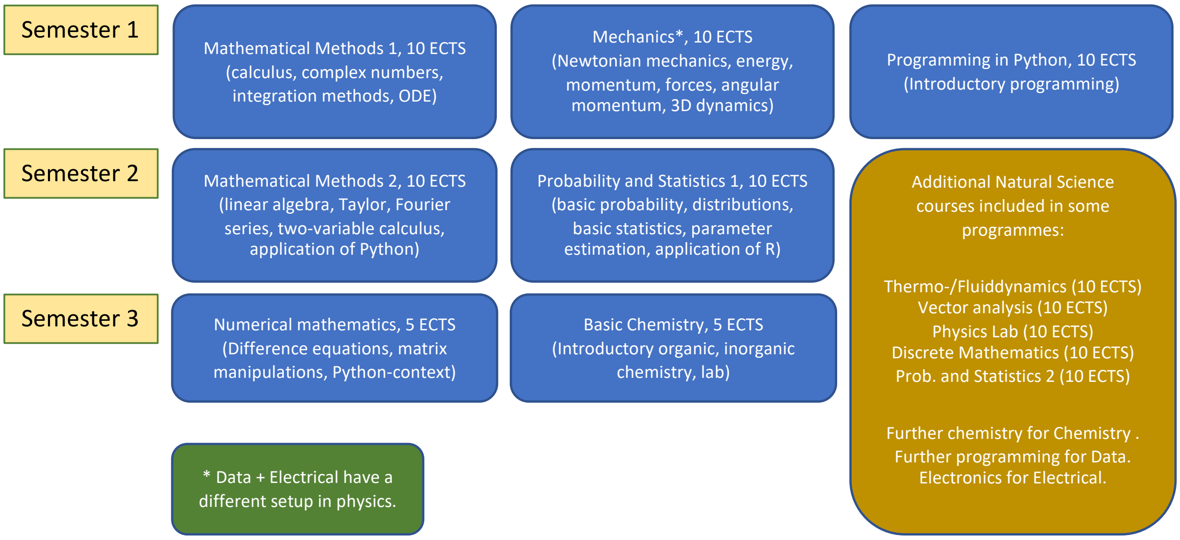
Implementation

This section exemplifies the development of study programs according to the Guidelines with emphasis on mathematics, from two different institutions. These show the flexibility at institutional level but comply with the national requirements.

Implementation, University of Stavanger

The University of Stavanger, UiS, offers 3-year engineering degrees in Chemistry and Environment Engineering, Data Technology, Automatization and Electronics design (Electrical), Machine Engineering, Building Engineering, Energy and Petroleum Technology and Energy- and Geoscience.

At the Faculty level, the institution has opted for stand-alone, dedicated courses, jointly for (almost) all degree programs, and all programs are explicitly designed to qualify for the Siv. Ing. title. The basic format is displayed as a figure below (details vary between programs). Note that by default the course size is 5 or 10 ECTS, and that the ECTS-count for Statistics (10 ECTS), Physics (10 ECTS) and Programming (10 ECTS) is chosen to be larger than the minimal requirements for the Siv. Ing. title.



Joint courses have the disadvantage that they become very large (4-600 students). This provides for some unique pedagogical, didactical, and logistical challenges, involving the lecturer, many teaching assistants, many smaller problem session groups, exam office, study administration, and inter-departmental negotiations. The advantage is that all engineering students really are taught the same basic skills and acquire the same background in natural sciences, programming and mathematics. This for instance allows for mobility between study programs, as well as cross-disciplinary use of elective courses later on. There is also an economical aspect.

Implementation, University of South-Eastern Norway

The University of South-Eastern Norway (USN) was established in 2018 through a merger of Buskered University College (HiBu), Vestfold University College (HiVe) and Telemark University College (HiT).

At HiBu, basic maths courses were taught by physicists and mathematicians. The institution has for many years contributed to the national bodies in charge of defining the maths content in higher education (Universities Norway, the Norwegian mathematical council). As the lecturers were responsible for defining the curriculum in their subjects, the national Guidelines were adhered to without discussion. It was understood that the main goal of the courses was that the students should understand the basic theory behind the calculations, and that they should be able to implement this understanding to their own engineering specialisation.

At HiVe, basic maths was taught by physicists rather than mathematicians. They were less eager to follow the Guidelines to the letter but applied a more flexible interpretation. However, the result was not very different from that at HiBu, perhaps with slightly more focus on solving exercises than on proving the theorems.

HiT was a more rural college and struggled with recruitment of students. They concentrated on vocational trained students (Y-vei) and made their own curriculum suited for those students. The math courses were designed without reference to the Guidelines and focusing on computational mathematics, emphasizing Python-programming of standard algorithms.

Implementing the Regulations at the new combined institution USN required some negotiation and consensus on the common curriculum and the plans for the different courses. The result is a combination of the curriculums of HiBu, HiT, and HiVe, fully compliant with the Guidelines, but with the addition of Python-programming of all the explicit algorithms stated in the Guidelines.

The conclusion is that the Guidelines are implemented in a reasonable way, but with a substantial element of programming, which does takes time away from mathematics understanding. Still, the scientific staff is very fond of the Guidelines, since they are helpful in disputes on the design of courses.

Discussion and Reflections

The Norwegian Engineering education is defined through Regulations, which are mandatory, and Guidelines anchored in the regulations. Because of flexibility within these guidelines, they may possibly be perceived as recommendations. For the title Siv. Ing, additional requirements are adopted.

The stated intention of establishing a national standard, notably for the content of natural sciences and maths, relies on the Guidelines providing workable definitions of what this standard should be, both in terms of learning outcomes, volume (ECTS count) and level.

The content and formal status is a subject of constant debate and development between the Ministry, UHR and management and scientific staff at the institutions offering engineering programs.

At one end of the spectrum, there are those who insist that to be an engineer, one must have a minimal knowledge of all the natural sciences and maths (as defined above), irrespective of specialisation, and irrespective of which institution one trained at.

At the other end, there are those who believe that engineering programs are best designed locally, keeping in mind the Guidelines but with the option to deviate from them at the discretion of the institution itself. And in particular that natural sciences and maths should enter strictly as tools, to the extent that they support a given engineering specialisation.

In the discussion, all positions in-between are represented, and in the regular revisions of the Guidelines a compromise (though rarely a consensus) is reached. This time around, the recommendations in mathematics are very clear and concise, but since the Guidelines as a whole are not perceived as mandatory, it is unclear whether a common standard may be achieved.

In one of the examples of implementations described above (UiS), the requirements for the Siv. Ing. title effectively ensure that the Guidelines are followed. But for institutions who do not offer master programs who gives the Siv. Ing. title, there is no enforced standard in natural sciences and maths except the recommendations in the guidelines.

In the other example implementation (USN), the Guidelines were used successfully as three engineering programs were merged into one as part of an institutional merger. If strong local leadership insists that the Guidelines must be adhered to, they will still be valuable tools to coordinate engineering education.

References

Burkhard Alpers et. Al (2013). A Framework for Mathematics Curricula in Engineering Education A Report of the Mathematics Working Group. A Framework for Mathematics Curricula in Engineering Education – SEFI

Jakobsen, M.M: A competence-based framework for engineering education (2012). 40th SEFI Conference, 23-26 September 2012, Thessaloniki, Greece, Proccedings 40th SEFI Conference, publisher SEFI, Societe Europeenne pour la Formation des Ingenieurs.

Jakobsen, Lurås, Nygård (2017). Evaluation of Implementation of New Framework Regulations for Engineering Education in Norway. Proceedings of 45th SEFI Conference, 18-21 September 2017, Azores, Portugal. https://www.sefi.be/wp-content/uploads/SEFI_2017_PROCEEDINGS.pdf, p 878.

Kent, Ph. and Noss, R. (2002) "The Mathematical Components of Engineering Expertise: The Relationship between Doing and Understanding Mathematics." In *Proc. of the IEE 2nd Annual Symposium on Engineering Education*, London.

Ministry of Education and Research, Norway (2018). National Curriculum Regulations for Engineering Education

On the Understanding Mathematics

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It is often difficult to teach mathematics for understanding. Many students seem to prefer to learn computational skills by rote, and seem to remember very little of the course contents the following semester. The reasons for this are surely manifold. In this talk we review the concept of understanding, drawing on a wide range of ideas from philosophy as well as pedagogy and the didactics of mathematics. We argue the understanding we seek in engineering mathematics should not be the same as what we seek in a mathematics degree.

1 Introduction

Most educators agree that we need to teach for understanding, rather than for rote learning. When we teach a mathematics degree, we expect the students to understand the relationships between different concepts, and understand why a given statement is true, and why given techniques work. When, in contrast, we teach service mathematics for engineering or other degree programmes, it is not obvious that the students need understanding in this sense. Certainly, rather few students are motivated for it.

Given how difficult it appears to understand mathematics, we are often content, in service mathematics, if the students acquire a modest selection of computational skills. However, the German mathematics didactician Wolfram Meyerhöfer[1] claims that also students who struggle with mathematics can learn to understand, and moreover they *can only learn computational skills if* they understand.

Understanding seems to be sought, one way or another, by most people. As one of my students put it, «can't you just give us a formula, so that we can start to understand?» While this seemingly contradictory question puzzled me at the time, there may still be some sense behind it. In this presentation we shall review what we mean by understanding in the context of engineering mathematics, and consider possible implications for our teaching. This is a philosophical project, building analysing existing ideas from the philosophy of education and hermeneutics. Our contribution is the situation of these ideas in the engineering education.

2 On the Necessity of Understanding

While mathematics can be learnt both by rote and through understanding, rote learning has few, if any, defenders. The problem, says Holm (2012, p. 44), is that rote learning is fragmented. Each procedure becomes an isolated unit of knowledge. Selden, Selden, Hauk and Mason (2000) has documented that even high-performing students fail to solve non-routine problems, in spite of demonstrating all the prerequisite, constituent skills on routine problems. In other words, they know the procedures as isolated units of knowledge, applicable to routine problems, but they fail to access the same units in more complex contexts.

According to constructivist learning theory, long-term memory is facilitated by cognitive schemata, a term sometimes attributed to Jean Piaget. Instead of memorising fragments of knowledge, we integrate and organise new pieces of information in complex mental models, called schemata. It is easier to remember new information when we can make sense of it in relation to pre-existing knowledge. Thus we get the Matthew effect[2]. Those who have a lot of knowledge, in the form of cognitive schemata, can more easily gain new knowledge.

Schoenfeld (1988) relates the students' lack of problem solving ability to several common beliefs about mathematics. One of these beliefs is that mathematics problems can be solved quickly by anyone who understand the subject matter. As a corollary, if you do not see the solution quickly, there is no point in trying. Students with this misconception has not reason to engage in the trial and error which is critical to all kinds of problem solving (e.g. Simon, 1969, p. 97). This may be coupled with the belief that the successful mathematics students performs assigned tasks to the letter, as prescribed by the teacher, suggesting a limited number of ready procedures which can, in principle, be learnt by rote. More recent authors (e.g. Piercey & Militzer, 2017) have suggested to counter-act these misconceptions through inquiry based learning and similar approaches.

In this context, we can possibly understand the student who asked for a formula in order to understand. If the goal of the mathematics module to be able to solve the routine tasks set by the teacher, then a routine formula may suffice to let the student 'understand', or at least know, how to solve the tasks. If the more complex, non-routine problems are out of scope for the student's attention, then there may be no reason to seek further understanding. Many students are indeed able to learn the necessary skills by rote in order to succeed in the exam.

We should note that the students are very unlikely to solve routine problems by hand in their future careers. Routine solutions have long been computerised. It is the non-routine problems that still require human attention. Thus, there is no point in learning just the basic skills. The basic skills may be a step on the way, but it is only the students who proceed to comprehend more complex problems that will have any use therefor.

All of these observations support a focus on understanding in teaching and learning of mathematics, but they do not tell us much about what that means.

3 On the Nature of Understanding

Hermeneutics has received increasing attention as an approach to education in the last generation (e.g. Kemp, 2013; Kerdeman, 1998), and a main topic of hermeneutics is the study of *understanding*. However, 'understanding' takes different meanings in in hermeneutics and in epistemology (Kerdeman, 1998). In epistemology, understanding tends to refer to the mental process by which we gain knowledge about a concept, i.e. the process which not only gives us a fact, but tells us why we can be certain that the fact is true. A student who not only applies a given procedure, but also can *prove* that it is correct, is commonly considered to have understood the procedure.

In hermeneutics, as developed by Gadamer (2004) and others, understanding is a way of *being*. It is not a deliberate process, but a natural and necessary part of being human. Understanding is not what we know about the world around us, but rather *how* we relate to this world. Kerdeman (1998) describes understanding, and learning, as *transformational*, as opposed to acquisitional. It changes us so that we see our environment in a different way.

What bearing, then, does this hermeneutical concept of understanding have on the teaching and learning of mathematics? To answer this, let us consider for a moment what it means to *be* a mathematician, and also what it means to be an engineer. As mathematicians, we have a special way of *seeing* the world around us. We recognise in the patterns of the real world, mathematical shapes and relations. Based on this recognition we can model phenomena and use the models to measure and predict behaviour. This recognition is not necessarily a conscious process. Very often we recognise the key concepts and relations immediately, without thinking. It is just part of how we think as mathematicians.

In the *Framework for Mathematics Curricula in Engineering Education* 2013, SUFI formalises this as the modelling competency, following Niss and Højgaard (2011). The engineers that we train depend on being able to *see* mathematical relations in physical systems in order to build models. And conversely, in the their mathematical models, they need to *see* emerging technological devices and how they will work for human users. In the same way as architects see buildings and their functions in their sketches (Schön & Wiggins, 1992), engineers need to see engines and gadgets, and their functions, in their mathematical models.

In this sense, understanding of mathematics is clearly transformational. It is only when it changes the way in which they see, comprehend, and assess physical systems, that we can say that an engineering student has a useful understanding of mathematics.

4 On the Process of Understanding

Hermeneutics traditionally referred to the interpretation of texts, particularly legal and holy texts. The hermeneutic circle developed as an iterative methodology of interpretation, focusing alternately on the whole and the parts (sentences and paragraphs) of the text. After a review of the parts, the whole will be seen in a different light when it is reread (and vice versa).

Heidegger and Gadamer were the pioneers extending the concept of hermeneutics to understanding in general, and to learning. New understanding, says Gadamer (2004), springs from the tension between the parts and the whole, or between the strange and the familiar in a continuous and iterative hermeneutic process. When we consider new pieces of information (parts) we always interpret them in light of our pre-existing understanding of the whole, i.e. our cognitive schemata in Piaget's terminology. Sometimes there is a conflict, a tension, as the new information appears inconsistent with our pre-understanding. This conflict forces us to reinterpret what we know, and then there is learning.

Prior understanding is essential for learning. Without it, we end up in Meno's paradox from Plato's famous dialogue. If we know, there is nothing to learn. If we do not know, we cannot know what to look for, and thus not learn it. This, however, considers knowledge as a Boolean property, where we either know or not. Understanding, in contrast, is partial, and it is ever-changing as we interact with the world.

Understanding, in this sense, is existential. Gadamer speaks of «lived» understanding. We are not talking of compartmentalised knowledge which can be applied or ignored depending on context, but of a critical part of our life and understanding thereof. When we encounter tension between the familiar and the strange, it challenges us at a personal level, and it cannot easily be ignored.

5 On Lived Mathematics

Both the hermeneutic argument of Gadamer and the constructivist argument of Piaget lead to the same conclusion. Efficient learning depends on pre-understanding. The question is, how does this apply to mathematics teaching.

When mathematics is taught for understanding in the epistemological sense, knowledge is constructed deductively. Previous knowledge is treated as axioms from which new knowledge is deduced. For some students this works well. Even the abstract axiomatic system can make a lived experience for the student who takes a genuine liking therein and spend enough time deducing and exploring logical implications. This is not the typical engineering student, though. The thesis we argue is that these students have *other* lived experiences from which they can construct a sound understanding of mathematics.

A student commencing on an engineering degree is likely to have some understanding of the machinery they want to learn to design. The understanding may be immature, naïve, or even a dream, but it is an essential part of the being an engineer *in spe*. If, in this context, we introduce mathematics as a way of *seeing* the machinery, there is transformational potential. The student can change the naïve perspective of a user or a dreamer, into that of an engineer who comprehend and craft technology. This gives both a foundation and a motivation for new knowledge.

This approach places the modelling competency in the centre, with double benefit. Firstly, it ensure relevance, since mathematical modelling is a core task in most engineering disciplines. Secondly, by modelling familiar systems from the real world, we ensure that the students have relevant cognitive schemata to use as a starting point. This is not always the case when mathematics is built deductively from previous course content. Even if they have passed the exam, there are students who are not fully familiar with pre-requisite mathematics.

6 On the Coverage Issue

Teaching for understanding, we have to face the *coverage issue* (Yoshinobu & Jones, 2012). The mathematics syllabus is often laden with a long list of knowledge outcomes, and time barely suffices to tell the students what they need to know. There is no time left to discuss what we understand.

Excessive coverage is harmful. Whitehead (1967, p. 1) points out that schools of learning with ferment of genius in one generation, often degraded into pedantry and routine. The reason, he says, is that they are overladen with *inert ideas*, that is ideas which are merely received into the mind without being utilised. Part of the cure prescribed by Whitehead (1967, p. 3f) is proof; not the epistemological proof that the idea is true, but a proof of 'worth'. After all, this is how we usually approach new knowledge when we are not subject to the discipline of school. First, we appreciate its worth, and if it is worth knowing, we may seek confidence through a proof of truth. Engineering mathematics proves its worth when it improves our understanding of engineering systems.

Conservative approaches to mathematics teaching is based on the argument that 'the mind is an instrument, you first sharpen it, and then use it' (Whitehead, 1967, p. 6). Mathematics teachers try to sharpen the students by drilling them in basic skills, and then leave for engineering teachers to teach them how to make use of it. This is a misconception, because it is not how the mind works. The mind is never passive, always engaged in a hermeneutic circle to make sense of new and old information. On this both Gadamer and Whitehead agree.

Teaching mathematics as an isolated subject, even if fruitful for some students with a keen interest, runs two serious risks. Firstly, it is more difficult to remember and to understand a subject out of context. Only students who can establish a relevant context of their own are likely to succeed. Secondly, it disconnects different bodies of knowledge, which were supposed to support each other. Mazur (2009) quotes a student asking,

> *How should I answer these questions? According to what you taught me or according to the way I usually think about these things?*

Instead of giving different tools to gain a richer understanding of reality, we risk developing competing understandings of reality.

7 On the Freedom to Learn

We are not arguing that mathematical knowledge, in the conventional sense, should not be taught. It is a question of balance. The students need knowledge, but to remember the knowledge, they need time and freedom to incorporate it into their understanding. Whitehead (1967) elaborates on this balance. Through discipline, we can pump up the students with knowledge, but without the freedom to explore and develop the knowledge, the ideas become inert.

Rogers and Freiberg (1994) pointed out the paradox of school discipline. In working life, we generally succeed by being unique individuals, through original ideas and innovation. Many taught courses value students who do exactly as they are told, and top grades are given for solving the same problems as everybody else, using standardised solution techniques. While this was useful skill for engineers who started their career as *computers* in 1950, it is not what the world will ask from our students in the 21st century.

Freedom means both the freedom to use one's own experiences as the anchor point for topics studied, and to prioritise and select (some) topics according to personal interest. Inevitably, some topics may have to be skipped, or considered very briefly, to give time to process the topics of greatest worth. This is not necessarily a problem. Pure knowledge is easily forgotten and easily looked up in reference works. What takes time to learn is to see and think mathematically, or what Whitehead called *wisdom*, Aristotle called *techne*, and Polanyi called *tacit knowing*.

8 Conclusion

We have sought to improve our understanding of what understanding entails in mathematics. As far as the engineering education is concerned, a good understanding of mathematics as an independent discipline is neither necessary nor sufficient, and for many students it is unnecessarily hard to achieve.

For the engineer (among others) mathematics is a way to see, comprehend, and manipulate physical systems. Mathematics is not understood before it leads to a change in how we *see*.

Teaching mathematics as an independent subject, purely on the subject's own terms, forces the student to memorise a lot of knowledge with little opportunity to relate it to their own lived experience. This makes it hard to remember, and we cannot expect the students to retrieve all the mathematical knowledge when they get to study application in a later module. The hermeneutic circle depends on short iterations alternating between the familiar and the strange. Dedicating each taught module to either the familiar or the strange gives too long iterations, with too much information to be memorised before it is processed.

The problem we meet when we teach mathematics as a purely abstract discipline, is that we ask the students to build a completely new system of knowledge, detached from their lived experience. The separation of two systems of knowledge dissolves the tension between familiar and new ideas, which should have been the source of new understanding. Without the benefit of relevant, lived understanding, the fundamental mathematics becomes harder to understand, and even impossible for some.

Therefore, I propose that, in the context of service mathematics, understanding should mean *to understand real world problems and situations in terms of mathematics*. This is essentially the modelling competency in Mogens Niss' framework. However, we hold that this understanding is not a separate learning outcome to be pursued after basic training in mathematics. Rather, it is, except for students with a genuine prior interest and pleasure in mathematics in its own right, a necessary part of learning basic mathematics.

Notes

[1] «Ich will, dass jedes Kind rechnen lernt», *Der Spiegel*, 27.10.2013

[2] Often quoted from Mat 25.29, but it also appears in Mat 13.12f, in the context of learning and understanding.

References

Gadamer, H.-G. (2004). *Truth and method* (Second, Revised). Translated by Joel Weinsheimer and Donald G. Marshall from *Wahrheit und Methode (1960)*. Continuum, London, New York.

Holm, M. (2012). *Opplæring i matematikk*. Cappelen Damm.

Kemp, P. (2013). *Verdensborgeren: Pædagogisk og politisk ideal for det 21. århundrede* (2. reviderede udgave). Hans Reitzels Forlag.

Kerdeman, D. (1998). Hermeneutics and education: Understanding, control, and agency. *Educational Theory*, *48*(2), 241–266.

Mazur, E. (2009). Farewell, lecture. *Science*, *323*(5910), 50–51.

Niss, M. & Højgaard, T. (2011). *Competencies and mathematical learning: Ideas and inspiration for the development of mathematics teaching and learning in Denmark*. IMFUFA, Roskilde university.

Piercey, V. & Militzer, E. (2017). An inquiry-based quantitative reasoning course for business students. *PRIMUS*, *27*(7), 693–706.

Rogers, C. R. & Freiberg, H. J. (1994). *Freedom to learn* (Third). Macmillan College Publishing Company.

Schoenfeld, A. H. (1988). When good teaching leads to bad results: The disasters of 'well-taught' mathematics courses. *Educational Psychologist*, *23*(2), 145.

Schön, D. A. & Wiggins, G. (1992). Kinds of seeing and their functions in designing. *Design Studies*, *13*(2), 135–156. doi:https://doi.org/10.1016/0142-694X(92)90268-F

Selden, A., Selden, J., Hauk, S. & Mason, A. (2000). Why can't calculus students access their knowledge to solve non-routine problems. *Issues in mathematics education*, *8*, 128–153.

Simon, H. A. (1969). *The sciences of the artificial* (1st). MIT press.

Whitehead, A. N. (1967). *The aims of education*.

Yoshinobu, S. & Jones, M. G. (2012). The coverage issue. *Primus*, *22*(4), 303–316.

Mathematics in a Programme for Electronic Systems Design and Innovation

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Abstract

This paper presents a preliminary report from the project Mathematics as a Thinking Tool, where mathematics and electronics are taught in close collaboration between the teachers in the two fields. One goal of the project is that students should experience how and why mathematics is relevant for them in their engineering specialisation. Another, more long-term goal, is to develop an approach driven by the learning goals of the study programme rather than learning goals in specific subjects. Examples are given to show how mathematical theory is useful for justifying principles in electronics, thereby helping students to acquire a deeper understanding of these principles. Some results from student surveys will also be included.

Introduction

In engineering education, the tension between theory and practice, between academic and professional aims, has roots going far back in time (Edström, 2018), and recent studies show that this tension still persists (Carvalho & Oliveira, 2018). Although mathematics has from early on been regarded as an important subject in engineering, part of the tension concerns precisely the role and perceived relevance of mathematics (Flegg et al., 2012; Gueudet & Quéré, 2018). Also, the question of what kind of mathematics should be taught to engineers and who should teach it, has a long history (Alpers, 2020; Bajpaj, 1985). In our project, the aim is to develop a close connection between mathematics and engineering subjects early in the study programme, while maintaining conceptual understanding in both fields. The overarching idea of the project is formulated as developing mathematics as a 'Thinking Tool'.

Ideas about mathematics as a thinking tool in engineering are also not new. Already in 1985, Scanlan, in a talk about mathematics in engineering education, concluded by stating that mathematics should be an essential part of the students' formation and "not a set of 'tools' to be acquired before proceeding to the 'important' part of the course" (Scanlan, 1985, p. 449). However, it is well documented that the challenges with applying mathematics that students are supposed to have learned, when they need it in the engineering courses still persist (Carvalho & Oliveira, 2018; Harris et al., 2015).

In recent years, the so-called CDIO Initiative (Crawley et al., 2014) has gained momentum in engineering education. A crucial point in the CDIO approach is that conceptual understanding, rather than memorisation of facts and definitions, or the simple application of a principle, should be the aim of the education. In addition, the CDIO approach values *contextual learning,* in the sense that new concepts should be presented in situations that students recognise as important to their current and future lives (Crawley et al., 2014, pp. 32-33).

A pilot project in electrical engineering

At NTNU, engineering education at master's level is mostly offered as five year integrated programmes. During the first two years, all students take four courses of in total 30 ECTS of mathematics and one course of 7.5 ETCS of statistics and probability. These five courses are taught with largely the same content for all engineering programmes. They comprise standard topics like single and multivariable calculus, linear algebra, differential equations, vector analysis, Fourier analysis, Laplace transforms, some complex analysis, and statistics and probability. These are topics common to most engineering programmes around the world and have been so for several decades.

Starting in 2020, a pilot project aims at investigating how mathematics can be learned in a way that is more integrated with other teaching and learning activities of a particular programme. The pilot concentrates on the programme *Electronic System Design and Innovation,* and it involves developing four new mathematics courses (one for each of the first four semesters) and five engineering courses that run in parallel. Thus, there is always at least one engineering course taught in the same semester as each mathematics course.

An important prerequisite for the pilot is the attitude that integrated teaching and learning of mathematics for engineers is a shared responsibility between those who teach mathematics and those who teach applications. Another requirement is that the topics covered in the courses should largely be the same as for other engineering programmes. The emphasis and sequencing, however, are adapted to fit the engineering programme in question.

The planning and execution of the pilot is thus a collaboration between teachers from Department of Mathematical Sciences and Department of Electronic Systems, as it involves restructuring of both existing engineering courses and the development of the four mathematics courses. Accordingly, joint planning of teaching and learning activities is done in advance for all involved courses, and involved teachers are continuously communicating during the semester.

Three cases

A basic idea in the project is that learning of both engineering and mathematics will benefit from a coordinated approach. We will now present three examples of how this has been attempted during the first year of the project.

Nonlinear behaviour

Circuit theory is traditionally taught early in study programmes in electrical engineering. It is a topic of strong traditions, with limited variation in content, teaching, and learning activities, both over time and across universities. Traditionally, emphasis has been placed on linear circuits, and hence most problems have been solvable using elementary algebra. As a consequence, all problems have analytic solutions, in accordance with a young student's expectation of how a mathematical problem should behave.

In reality, most engineering problems are not linear, and analytical solutions cannot be found. In the project we have tried to introduce such problems early, in order to develop the students' mental habits so that they can cope with realistic situations more easily. Figure 1 a) shows the first circuit that the students get familiar with in the circuit theory course, a light emitting diode (LED) powered by a voltage source.

a) b)

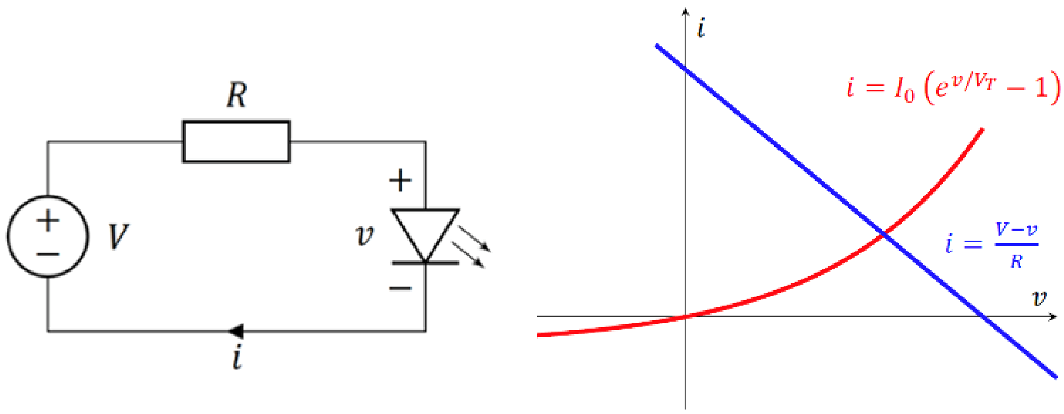

Figure 1 a) Circuit with LED. b) Graphical solution to set of equations (1) and (2).

Given the voltage V and the resistor with resistance R, the problem is to determine the current i through the circuit. To solve this problem, an expression for the diode current as a function of voltage v is needed. An adequate mode is given by Shockley's diode model,

$$(1) \qquad i = I_0\left(e^{v/V_T} - 1\right),$$

where I_0 and V_T are constants characteristic for the LED. The same current passes through the resistor, and Ohm's law gives

$$(2) \qquad i = \frac{V-v}{R}.$$

It is easy to show that there exists a unique choice of i and v that simultaneously solve the two equations, illustrated in Figure 1 b). However, closed expressions for i and v cannot be found, and in the mathematics course this example is used as an example of how even simple problems might require numerical solution techniques to be used.

Using a praxeological analysis (Bosch & Gascón, 2014), Rønning (to appear) has shown how analysis of an electric circuit requires an interplay between the mathematical praxeology and the praxeology from electrical engineering. Although the generating question for the problem, as well as the answer, comes from electrical engineering, subquestions, works to be studied, and partial answers (Chevallard, 2020), come from both praxeologies. The solution process therefore involves an interplay between the praxeologies, indicating that there is much to be gained from a close connection between the two subject fields.

The superposition principle

In circuit theory, the superposition principle states that in a linear circuit any voltage or current can be decomposed in different components, one for each independent source in the circuit. The component corresponding to one source can be calculated by setting all other sources equal to zero. Then the complete solution is obtained by adding the currents. This result is stated in all elementary textbooks on circuit theory (e.g., Nilsson & Riedel, 2011), but usually without proof or further justification. The reason is probably that elementary circuit theory is usually taught before linear algebra, and one can therefore not assume that all students are familiar with the necessary mathematics for the justification, which follows from the linear nature of Ohm's and Kirchoff's laws. In the project we have deliberately introduced linear algebra very early, so that the teacher in the circuit theory course can actively use results from linear algebra to explain why the superposition principle holds. This is particularly appreciated by those students who want to dig deeper and understand *why* the tools work, not only how to use them.

Parseval's identity and distortion

Fourier analysis is a topic with high relevance for the electronic systems designer as well as for other types of electrical engineers. Consequently, this is part of the mathematical foundation in all EE programs around the world. What often happens, however, is that students do not recognise the concepts from mathematics when the context is switched to engineering, where notation and vocabulary may be entirely different (Rønning, 2021). When, in addition, the two contexts are separated in time, possibly by years, engineering instructors often end up by teaching Fourier analysis from scratch once more "in their own way", without reference to students' earlier experiences in mathematics courses, thus obliterating any potential connections.

In this project, Fourier analysis is taught in parallel in mathematics and engineering courses. In the second semester, Fourier series are taught in the mathematics course while impedances, filters and frequency responses are taught in the engineering course. In contrast to earlier approaches, the connections to linear algebra, projections and inner products are explained. The relevance of Parseval's identity is demonstrated during a design project in which students have to generate a sinusoidal signal with a particular frequency. Students can identify and quantify the magnitude of unwanted harmonic components by a spectrum analyser and use Parseval's identity to compute the *Signal-to-Distortion-Ratio* as a quantitative measure of the quality of their design.

Although Fourier series is an important topic both in mathematics and in electrical engineering, their motivation is slightly different in the two fields. In mathematics, the main interest is in representing periodic functions in a Fourier series and study convergence properties of this series, whereas in electrical engineering, the main interest is in the Fourier coefficients (the spectrum of the signal), and not so much in the series (Rønning, 2021).

Student survey

To study the effects of the pilot project, two student surveys have been performed, in the first (n=40) and second (n=40) semester. A survey from 2013 (n=662) given in the first semester mathematics course to students from all five-year integrated engineering study programmes is used for comparison. The number of respondents of the surveys is given in the parentheses. The students were asked on a four-point Likert scale to indicate to what degree they agreed with the statements (1) "I am eager to understand concepts and underlying principles in the course" and (2) "I have an understanding of why mathematics will be important for me later in my education".

Comparing the survey from 2013 with the one from the pilot project, only small differences can be observed. In 2013, 92% of the respondents indicated that they somewhat or totally agreed with statement (1), compared to 85% and 88% for the first and second semester of the pilot project, respectively. For statement (2), 85% of the respondents somewhat or totally agreed in 2013, compared to 100% and 98% for the first and second semester of the pilot project, respectively.

In open-ended answers on the survey from the first semester of the pilot project, differential equations and circuits, and the superposition principle are explicitly mentioned as topics where the students see a close connection between mathematics and the engineering course.

The high number of respondents that agree with the statements in 2013, makes it difficult to see clear effects of pilot project. As none of the students have experienced both types of mathematics courses, it is difficult to know what kind of benchmark and priors they are using when answering the questions. A different approach, for example interviews or reflective diaries, is needed to further explore the effect of the pilot project.

Discussion

The preliminary experiences from the pilot project seem to indicate that there is much to be gained by seeing mathematics and engineering subjects in connection, and that this connection to a large extent can be obtained by a mutual adjustment of the subjects in question, without compromising their distinctive characters. There is evidence to indicate that the close contact and collaboration between the subjects mutually support learning in both fields, in line with ideas of contextual learning (Crawley et al., 2014). Student surveys also indicate that the students are able to pinpoint particular topics from the subjects that are mutually supportive.

There are plans for extending the project to cover also other study programmes in engineering and work will commence to analyse what adaptations are relevant for individual study programmes. At NTNU, there are in total 17 study programmes in engineering education at master's level, so a crucial question is how many different versions of mathematics are needed to cover all programmes in a good way.

References

Alpers, B. (2020). *Mathematics as a service subject at the tertiary level. A state-of-the-art report for the Mathematics Interest Group.* Brussel: European Society for Engineering Education (SEFI)

Bosch, M. and Gascón, J. (2014) "Introduction to the Anthropological Theory of the Didactic (ATD)." In A. Bikner-Ahsbahs & S. Prediger, eds. *Networking of theories as a research practice in mathematics education,* Cham: Springer, 67–83.

Carvalho, P. and Oliveira, P. (2018) "Mathematics or mathematics for engineering?" In *Proceedings from 2018 3rd International Conference of the Portuguese Society for Engineering Education (CISPEE).*

Chevallard, Y. (2020). Some sensitive issues in the use and development of the anthropological theory of the didactic. *Educação Matemática Pesquisa*, 22(4): 13–53.

Crawley, E. F. et al. (2014). *Rethinking engineering education. The CDIO approach* (2nd Ed.). Cham: Springer.

Edström, K. (2018). Academic and professional values in engineering education: Engaging with history to explore a persistent tension. *Engineering Studies,* 10(1): 38-65.

Flegg, J. et al. (2012). Students' perceptions of the relevance of mathematics in engineering. *International Journal of Mathematics Education in Science and Technology,* 43(6): 717–732.

Gueudet, G. and Quéré, P.-V. (2018). "'Making connections' in the mathematics courses for engineers: The example of online resources for trigonometry." In V. Durand-Guerrier et al., eds. *Proceedings of INDRUM 2018. 2nd conference of the International Network for Didactic Research in University Mathematics,* Kristiansand: INDRUM, 135–144.

Harris, D. et al. (2015). Mathematics and its value for engineering students: What are the implications for teaching? *International Journal of Mathematical Education in Science and Technology,* 46(3): 321–336.

Nilsson, J. W. and Riedel, S. W. (2011). *Electric circuits* (9th Ed.). Upper Saddle River, NJ: Pearson Education Inc.

Rønning, F. (to appear). Learning mathematics in a context of electrical engineering. In R. Biehler et al., eds. *Practice-oriented research in tertiary mathematics education: New directions,* Springer.

Rønning, F. (2021). The role of Fourier series in mathematics and in signal theory. *International Journal of Research in Undergraduate Mathematics Education.* https://doi.org/10.1007/s40753-021-00134-z

Scanlan, J. O. (1985). The role of mathematics in engineering education: An engineer's view. *International Journal of Mathematical Education in Science and Technology,* 16(3): 445-451.

Three-Level System for Teaching Mathematics in Engineering Education

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Abstract

Mathematics is the foundation on which the engineering studies are built on. Therefore, it is important to transfer the skills and the understanding of mathematics to the students in engineering in the very beginning of their studies. These courses are commonly known as 'service math courses', and teaching these courses provides challenges. Selected educational approaches as solutions for these challenges will be presented in this paper. The teaching method is divided into three levels of weekly exercise material, as well as three tests that need to be covered. Comparisons to the results of the same course a year before without the three-level system will be discussed. We will report on heuristics and show some results of the trail phase of this format. Examples of each level will be presented. A statistical analysis of the results and the student's engagement will be performed.

Introduction

In this contribution we present a new approach to constructing teaching and learning to increase student's motivation and at the same time offer support for the needs of groups with different mathematical backgrounds. One of the most common obstacles lecturers face while offering the so called 'service courses' is that the students of engineering discard the introduction to higher mathematics as not interesting, and partially unnecessary since the topics are not closely related to the subject of their studies. The second obstacle is that these courses are bound to deliver very complex materials in a short time and make sure that the content is understandable for the undergraduate students, which can be challenging at times.

We focus on the introductory mathematics course offered to students of physics, electrical engineering and geodesy and geoinformation. This will further be referred to as Mathematics 1. This course consists of lectures and complementary exercise sessions. In keeping with the concept of constructive alignment, the aim of the lecture is to impart definitions, concepts and relations between them, while during the exercise course students have to apply the knowledge they have acquired. The set of requirements for passing the exercise session course consists of weekly assignments and tests during the semester (Mid-Terms). The lecture course is completed by a final exam. The exercise materials, together with some materials for visualisation in the lecture, are done in Möbius which is supported by Maple.

Due to the COVID-19 pandemic, the importance of distance learning increased during the last year which made the students engage more with the electronic learning materials, and we can see grade improvement in the summer term 2020.

Structure of the course and first observations

Mathematics 1 in the summer term is a course that can be attended by the students of three different majors, and must be suited to their different interests, as well as different focus of their studies. The course is designed to be in three levels, in order to support students from different backgrounds and provide continuous assessment, as well as a lot of exercise material. The three – level assignments are provided on a weekly basis.

The first level must be completed by every student, and they need to score at least 90% of the available points. These examples are providing information and preparation on the topics and materials they should already know in order to follow the current material in the class. It consists mostly of problems and topics that are already, or should have been covered in school, or later during the term, in previous lectures. They can repeat these assignments as often as they want in a given time frame, and their best submission is graded. Each time they start an assignment again, they get a randomised version of the assignment, which provides necessary basic exercise material that is otherwise not available.

Only after they have completed the first level and refreshed or learned the necessary topics for the current exercise session, can they start with the regular exercise, or the second level. This is a traditional exercise, that is also covered in exercise sessions. Here, they have two attempts and need to obtain at least 60% in order to pass the course. These examples are given as homework and are also presented in by the students in a weekly exercise session. This requires deeper understanding of the topic and prevents the students from cramming the very broad material of this course short before an exam. Also, multiple solutions are discussed and explained, as well as some underlying questions that are clarified by a teaching assistant. After the start of the COVID-19 pandemic, these courses switched to online sessions with the use of Zoom. In summer term 2019 the students would present their solutions on a blackboard, whereas in 2020 they had distance-learning and presented their written solutions over a screen share. This unfortunately reduced the attention of the students that were not presenting, but on the other hand, more questions were asked, and the attendance increased. As already mentioned, they do not need to solve all of these examples in order to pass the class, but it is necessary to connect the skills they already obtained by solving the questions of the first level and new skills gained from the lecture. For each chapter covered in the lecture, a set of randomised exercises is provided as a practice for the test.

The third level is the so-called master class, and two to three examples are provided to the students each week, where a certain idea is presented, and they need to construct their own examples. Those problems are used to broaden the understanding of the subject and provide connections between seemingly different topics of the course. The students can gain bonus points if they solve these examples correctly, and this is not obligatory. In the next section, we will compare the final grades of the students who attempted these master class examples at least once, with the students who did not.

We will also observe some results from the summer term 2019 and compare it to 2020, to make a statement about the impact the exercise course has on the examination of the lecture. In 2019 the three-level system was not introduced yet, and the students only had weekly homework, which corresponds the current level two.

Our main goal in this system is to close the gap between the students of different backgrounds by providing enough exercise and preparation material for each of them, so they can successfully pass the course.

Presentation of Data

In this section grade distributions of the exercise session and the lecture are presented. In the summer term of 2020 particular attention is paid to the grade distribution between students who submitted at least one example of the last level and students who did not participate in the master class.

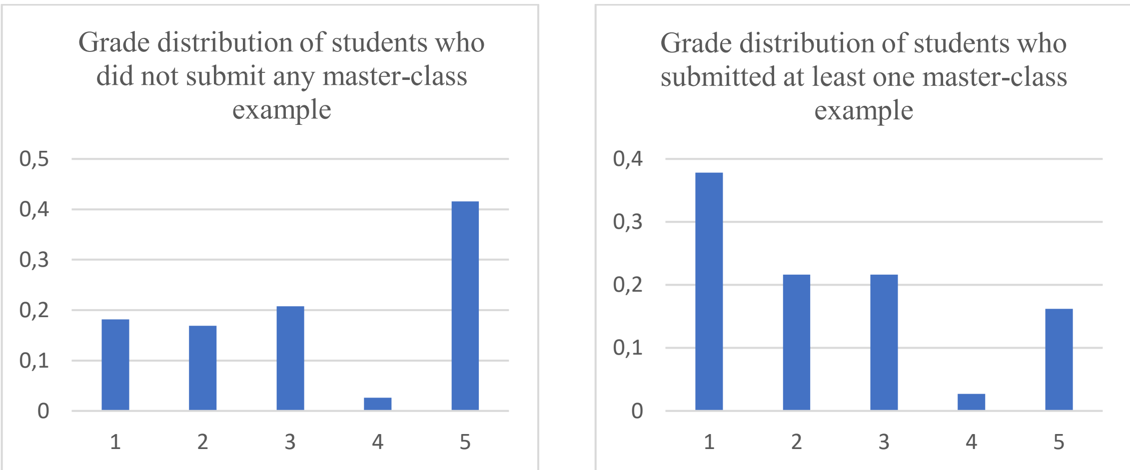

Figure 1. Grade distribution of students who did and did not participate in the master class in summer term 2020 in percentage terms.

In Figure 1 two histograms showing the grade distribution of the exercise session in 2020 are presented. In this term 37 students participated in the last level of the three-level system while altogether 114 students attended the exercise session. In this context a student is counted as a participant of this level if she/he submitted at least one example of the masterclass and presented it in the accompanying exercise.

The two histograms show that students who attempted last level examples showed significantly better results of the overall grade of the exercise session. About 37.8% of these students finished the course with the best grade, while only about 18.2% of those who did not attempt these examples managed to achieve the highest grade. The percentage of students who failed the course and did not participate in the master class is about 41.6%, which is more than two and a half times the percentage of those who did so. The share of grades between the best and worst grade is almost the same in the left and the right diagram of Figure 1. For example, the percentage of students who got the grade 4, which counts as sufficient, is about 2% in both categories.

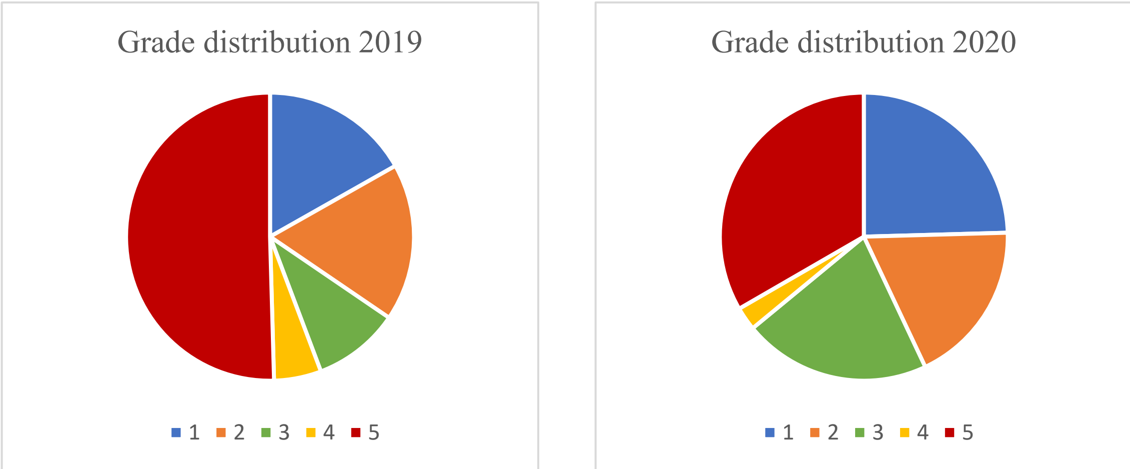

Figure 2. Grade distribution of the summer semesters of 2019 and 2020

Figure 2 shows the grade distribution of the summer semester of 2019 and 2020. This is of interest because in 2020 the three level-system of teaching mathematics was first introduced in engineering courses. In 2019 there were 113 students participating in the course, in 2020 there were 114. As we can see in the left diagram in Figure 2 the share of students, who failed the course, meaning they finished with the grade 5, is about half of the participants. The right diagram shows a smaller red region, in 2020 only one third of the participants could not successfully complete the exercise course. In contrast to 2019, where only about 17% finished the course with the highest grade, in 2020 this share is about a quarter of the students participating. The percentage of students, who finished the course with the second-best grade, is almost equal in both years with a total number of 20 students in 2019 and 21 students in 2020.

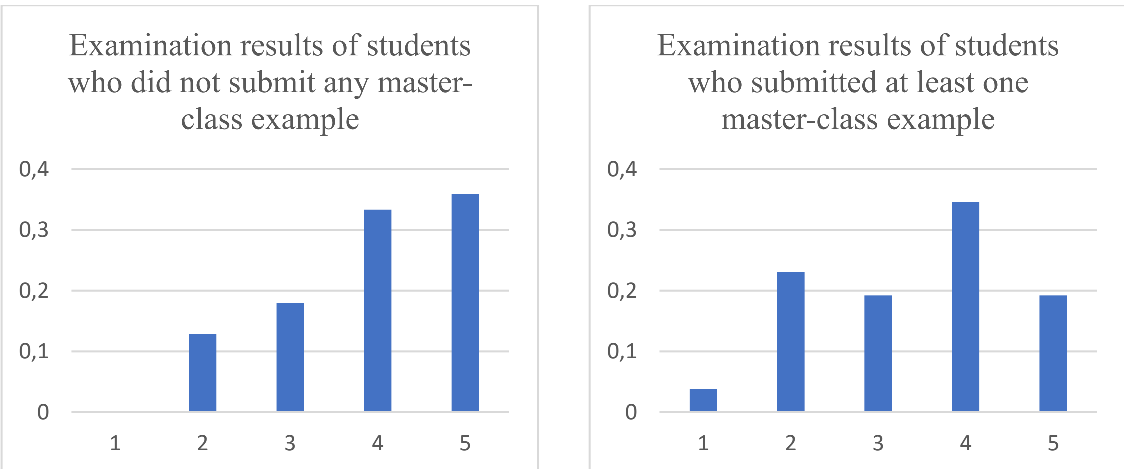

Figure 3. Grade distribution of the examination of the lecture of students who did and did not participate in the master class in percentage terms.

Altogether 65 students, who participated in the exercise session in 2020, took the exam of the accompanying lecture at least once. Figure 3 shows the grade distribution of the examination results of those students, divided into participants and non-participants of the master class. For this study, if a student took the exam more than once, the best grade was considered. About 20% of students, who submitted at least one master class example, and about 35% of the other participants of the exercise session could not pass the exam of the lecture. The ratio of participants, who passed the exam with the best or second-best grade and attended the master class, is more than twice as big as the ratio of students who did not.

Discussion of Results

The histograms in Figure 1 and Figure 3 both show that students, who submitted at least one example of the master class, are more likely to pass not only the exercise course, but also the examination of the accompanying lecture. Especially the grade distribution of the exercise course clearly suggests that also a higher ratio of participants of the master class managed to finish the course with the best or the second-best grade. Concerning the grade distribution of the examination of the lecture it must be considered, that only about half of the participants of the exercise course in 2020 took the exam. This is not unusual, since the lecture and the exercise course must not be completed during the same semester.

To check the influence of the introduction of a three-level-system in mathematical courses on the grades of the exercise course, the grade distribution of the summer semester of 2019 and 2020 was compared. In contrast to 2019, where only half of the students finished the course with a positive grade, about one third of the participants could achieve a positive result in 2020. Figure 2 clearly suggests that the introduction of a three-level-system has a positive impact on the grade distribution of the participants of the exercise course.

Conclusion and outlook

Based on the results of this study, it seems that the presented three-level system influences the grades of the exercise course and the lecture that goes with it. While Figure 1 shows, that participants of the master class also managed to finish the course with better grades, Figure 2 points out, that the number of students who completed the course with a positive grade increased with the introduction of the three-level system. Figure 2 also suggests, that since the exercise course was split in different parts with a clear evaluation in 2020, the scoring was more comprehensible, and therefore more motivating, than in 2019.

As shown in Figure 3, students, who submitted at least one example of the master class, were more likely to pass the exam, than those, who did not participate in the last level of the system. Hence, it can be concluded, that the introduction of the three-level system does not only have a positive impact on the exercise course, but also on the accompanying lecture.

This three-level system was further developed in the summer term 2021, but since the semester is not over yet, there is no data to evaluate. The exercise materials in Möbius and clear communication of requirements for the course got positive feedback from the students, so it is planned to introduce similar methods to other courses taught by our group. One of these courses is Practical Mathematics for Technical Physics, which is to be done in the winter term 2021.

References

Tigwell, K. and Prosser, M. (2013) "Qualitative variation in constructive alignment in curriculum design." In *Higher Education 67:* 141–154 (2014)

Körner, A., Winkler, S., Leskovar, R. and Gorgas F. (2018) "Online-Komponenten der Lehre an der TU Wien." In *Tagungsband ASIM 2018, 24. Symposium Simulationstechnik*

Modiz, C., Gorgas, F., Winkler, S., and Körner, A. (2020) „Heuristische Untersuchung der Abhängigkeit von Übungen und Vorlesung für den Prüfungserfolg." In *ARGESIM Report 59:* 481-484

Roediger, H.L. and Butler, A.C. (2011) "The critical role of retrieval practice in long-term retention." In *Trends in Cognitive Sciences, Volume 15:* 20-27